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HO LE HUY PHUC

DEVELOPMENT OF NOVEL MESHLESS METHOD
FOR LIMIT AND SHAKEDOWN ANALYSIS
OF STRUCTURES & MATERIALS

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DEVELOPMENT OF NOVEL MESHLESS METHOD FOR LIMIT AND SHAKEDOWN ANALYSIS OF STRUCTURES & MATERIALS

MAJOR: ENGINEERING MECHANICS

Supervisors:
1. Assoc. Prof Le Van Canh
2. Assoc. Prof Phan Duc Hung

Reviewer 1:
Reviewer 2:
Reviewer 3:
Declaration of Authorship

I declare that this is my own research.
The data and results stated in the thesis are honest and have not been published by anyone in any other works.

*Ho Chi Minh city, 3rd August 2020*

PhD candidate

**HO LE HUY PHUC**
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*Ho Chi Minh city, 3rd August 2020*

PhD candidate

**HO LE HUY PHUC**
Abstract

The proposed research is essentially concerning on the development of powerful numerical methods to deal with practical engineering problems. The direct methods requiring the use of a strong mathematical tool and a proper numerical discretization are considered.

The current work primarily focuses on the study of limit and shakedown analysis allowing the rapid access to the requested information of structural design without the knowledge of whole loading history. For the mathematical treatment, the problems are formulated in form of minimizing a sum of Euclidean norms which are then cast as suitable conic programming depending on the yield criterion, e.g. second order cone programming (SOCP).

In addition, a robust numerical tool also requires an excellent discretization strategy which is capable of providing stable and accurate solutions. In this study, the so-called integrated radial basis functions-based mesh-free method (iRBF) is employed to approximate the computational fields. To eliminate numerical instability problems, the stabilized conforming nodal integration (SCNI) scheme is also introduced. Consequently, all constrains in resulting problems are directly enforced at scattered nodes using collocation method. That not only keeps size of the optimization problem small but also ensures the numerical procedure truly mesh-free. One more advantage of iRBF method, which is absent in almost meshless ones, is that the shape function satisfies Kronecker delta property leading the essential boundary conditions to be imposed easily.

In summary, the iRBF-based mesh-free method is developed in combination with second order cone programming to provide solutions for direct analysis of structures and materials. The most advantage of proposed approach is that the highly accurate solutions can be obtained with low computational efforts. The performance of proposed method is justified via the comparison of obtained results and available ones in the literature.
Tóm tắt

Luận án này hướng đến việc phát triển một phương pháp số mạnh để giải quyết các bài toán kỹ thuật, và phương pháp phân tích trực tiếp được sử dụng. Phương pháp này yêu cầu một thuật toán tối ưu hiệu quả và một công cụ rời rạc thích hợp.

Trước tiên, nghiên cứu này tập trung vào lý thuyết phân tích giới hạn và thích nghi, phương pháp được biết đến như một công cụ hữu hiệu để xác định trực tiếp những thông tin cần thiết cho việc thiết kế cấu mà không cần phải thông qua toàn bộ quá trình gia tải. Về mặt toán học, các bài toán được phát biểu dưới dạng cực tiểu một chuẩn của tổng bình phương các biến trong không gian Euclide, sau đó được đưa về dạng chương trình hình nén phù hợp với tiêu chuẩn để, ví dụ chương trình hình hít bậc hai (SOCP).


Tóm lại, nghiên cứu này phát triển phương pháp không lưới iRBF kết hợp với thuật toán tối ưu hình nén bậc hai cho bài toán phân tích trực tiếp kết cấu và vật liệu. Thế mạnh lớn nhất của phương pháp đề xuất là kết quả số với độ chính xác cao có thể thu được với chi phí tính toán thấp. Hiệu quả của phương pháp được đánh giá thông qua việc so sánh kết quả số với những phương pháp khác.
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<td>Euclidean norm.</td>
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<td>2D</td>
<td>Two-dimensions.</td>
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<tr>
<td>3D</td>
<td>Three-dimensions.</td>
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<td>BC</td>
<td>Boundary condition.</td>
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<td>BEM</td>
<td>Boundary element method.</td>
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<td>CCCC</td>
<td>Clamped-clamped-clamped-clamped (BC).</td>
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<td>CCCF</td>
<td>Clamped-clamped-clamped-free (BC).</td>
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<td>CFCF</td>
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<td>CPU</td>
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<td>FE</td>
<td>Finite element.</td>
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<td>Finite element method.</td>
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<td>IQ</td>
<td>Inverse quadric.</td>
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<tr>
<td>iRBF</td>
<td>Indirect/integrated radial basis function.</td>
</tr>
<tr>
<td>LMEA</td>
<td>Local maximum-entropy approximation.</td>
</tr>
<tr>
<td>LP</td>
<td>Linear programming.</td>
</tr>
<tr>
<td>MF</td>
<td>Mesh-free.</td>
</tr>
<tr>
<td>MFM</td>
<td>Mesh-free method.</td>
</tr>
<tr>
<td>MLPG</td>
<td>Meshless local Petrov-Galerkin.</td>
</tr>
<tr>
<td>MLS</td>
<td>Moving least square.</td>
</tr>
<tr>
<td>MRKPM</td>
<td>Moving Reproducing kernel particle method.</td>
</tr>
<tr>
<td>MQ</td>
<td>Multi-quadric.</td>
</tr>
</tbody>
</table>
List of Abbreviations

MQ-RBF  Multi-quadric radial basis function.
NEM    Natural neighbour method.
NNI    Natural neighbour interpolation.
P-D    Primal-Dual.
PDE    Partial differential equation.
PDEs   Partial differential equations.
PIM    Point interpolation method.
PU     Partition of Unity.
PUFEM  Partition of Unity Finite element method.
RBF    Radial basis function.
RBFs   Radial basis functions.
RBFNs  Radial basis function networks.
RKP    Reproducing kernel particle.
RKPM   Reproducing kernel particle method.
RPIM   Radial point interpolation method.
SDP    Semi-definite programming.
RVE    Representative Volume Element.
SCNI   Stability conforming nodal integration.
SFEM   Smoothed finite element method.
SOCP   Second-order cone programming.
SPH    Smooth Particle Hydrodynamics.
SSSSS  Simply-simply-simply-simply (BC).
VEM    Volume element method.
XFEM   eXtended Finite Element Method.
YL     Yield line.
Chapter 1

Introduction

1.1 General

Limit and shakedown analysis or so-called direct analysis are well-known as the efficient approaches for safety assessment as well as structural design. The objective of both analysis models is to determine the maximum load that structures can be supported under the effect of different loading conditions. While limit analysis is usually used for the structures subjected to instantaneous loads increasing gradually until the collapse appears, shakedown analysis is appropriate for the structures under repeat or cyclic loads. The best advantage of direct analysis is the ability to estimate the ultimate load without obtaining the exact knowledge of loading path.

Based on the bounding theorems, direct analysis results in an optimization problem, in which the unknowns to be found are the velocity vector of kinematic form or the stress vector of static form, or both velocity and stress vectors of mixed formulation. Owing to the complexity of engineering problems, the numerical approaches are required to discretize the computational domain and approximate the unknown fields. Various numerical schemes have been proposed in framework of direct analysis, e.g. mesh-based or mesh-free methods. Besides that, one of major challenges in the field of limit and shakedown analysis is dealing with the nonlinear convex optimization problems. From the mathematical point of views, the resulting problems can be solved using different optimization techniques using linear or nonlinear algorithms.

In addition, owing to the increasing use of composite and heterogeneous materials in engineering, the computation of micro-structures at limit state becomes attracted in recent years. Known as the innovative micro-mechanics technique, ho-
mogenization theory is such an efficient tool for the prediction of physical behavior of materials. The macroscopic properties of heterogeneous materials can be determined by the analysis at the microscopic scale defined by the representative volume element (RVE). The implementation of limit analysis for this problem is similar to one formulated for macroscopic structures. A number of numerical approaches for direct analysis of isotropic, orthotropic, or anisotropic micro-structures have been developed and achieved lots of great accomplishments.

Figure 1.1 illustrates the whole numerical implementation for limit and shakedown analysis of structures and materials.
1.2 Literature review

1.2.1 Limit and shakedown analysis

Theory of limit analysis was developed in early 20th century based on the elastic- or rigid-perfectly plastic material model to support the engineers evaluate the collapse load of structures. The early theories of limit analysis was given by Kazincky in 1914 and Kist in 1917, then the complete formulation of both upper-bound and lower-bound theorems was firstly introduced by Drucker et al. [1]. Latterly, Hill [2] proposed an alternative formulation using the rigid-plastic material model. The landmark contributions to the development of limit analysis belong to Prager [3]; Martin [4]. The significantly contributions to the application of limit analysis in engineering problems can be founded in works of Hodge [5–7], Massonnet and Save [8], Save and Massonnet [9], Massonnet [10], Chakrabarty and Drugan [11], Chen and Han [12], Lubliner [13]. Since then, the researchers concern not only theory aspect but also the application of limit analysis in practical engineering problems.

In reality, structures are usually subjected repeat, cycle or even time-dependent loading. As a result, the structures may collapse when the loads are lower than those determined using limit analysis formulation. That means limit analysis may fail to provide a proper measure of structural safety. In this case, shakedown analysis can be used. The first formulation of shakedown analysis theorem was expressed by Bliech in 1932, then the static and kinematic principles were generally proved by Melan [14] and Koiter [15], respectively, which are well-known as lower bound and upper bound approaches. Next, the first separate criterion of shakedown (the incremental collapse criterion) was formulated by Sawczuk [16] and Gokhfeld [17]. Konig [18] completed the theory with his work on the alternative criterion. The separated shakedown theory is based on the fact that two different types of failure modes cause the in-adaptation of structures. It suggests the use of different formulations in dealing with two corresponding load factors, see e.g. Koenig [18]. The extensions of classical theorems to more realistic structures have attracted in recent years such as: geometrically linear structures, elastic perfectly-plastic material models, quasi-static mechanical and thermal loading, temperature-independent mechanical properties, negligible time-dependent effects. Among them, hardening and non-associative flow rules have been studied by Maier [19], Pycko and Maier [20], Heitzer et al. [21]. Studies on shakedown problem under geometric non-linearity
can be found in works of Weichert [22], Weichert and Hachemi [23], Polizzotto and Borino [24]. Shakedown has been extended to composites in study of Weichert et al. [25], damaged and cracked structures in studies of Feng and Gross [26], Hachemi and Weichert [27], Belouchrani and Weichert [28]. Another important area concerning the effects of temperature on yield surface was carried-out by Kleiber and Konig [29], Borino [30]. Recently, Pham [31] pointed out that real engineering materials may not yield but may fail under high hydrostatic stresses. In that work, the author has proposed a modified shakedown kinematic theorem using a fictitious material that can yield in bulk tension and compression. Le et al. [32] demonstrated that under repeat or cyclic load, structures can be collapse by the rotating plasticity, a general form of alternating plasticity, incremental plasticity and instantaneous plasticity.

The only difference between limit analysis and shakedown analysis is the loading conditions applying to structures. Limit analysis considers structures under one vertices loading, whereas shakedown model takes into account structures under a loading domain formed by various vertices. Consequently, the size of shakedown problem is larger than limit ones. It is important to note that limit analysis is the special case of shakedown analysis when number of loading vertices reduces to one. Therefore, in general, two models are very similar. There are two issues when handling that problems: first, it is in need of a robust tool for solving the nonlinear yield functions; and second, it is necessary to develop an appropriate numerical method for the approximation of problems. The brief overview of historical development of related matters will be expressed in the following.

1.2.2 Mathematical algorithms

One of challenges in solving limit and shakedown problem is finding out an appropriate optimization programming. In whole history of direct analysis, a number of optimization tools have been developed. Linear programming (LP) is simplest and widely used owing to the allowance of solutions for large scale problems. The contribution to this field can be found in works of Anderheggen and Knopfel [33], Cohn et al. [34], Nguyen [35], Sloan [36]. LP is simple for the implementation, but the expected solutions may not be obtained due to the yield functions can not be exactly described. Overcoming this drawback, the non-linear yield surface is treated by the approximation of itself piecewise linear, see e.g. Maier [37], Tin-Loi
Then, existing optimization algorithms, such as the Simplex method or Interior-point methods can be applied. The disadvantage of this scheme is the highly computational cost caused by linearizing the yield functions.

The nonlinear yield functions can be directly used in nonlinear programming formulations by means of Newton-type algorithm, for which eliminating the linear or nonlinear constrains using Lagrange multipliers is an important step in solving the problems. Then, an unconstrained functional formulation can be dealt with using several iterative methods. Devoting to the development of such algorithms, it should refer to works of Gaudrat [40], Zouain et al. [41], Liu et al. [42], Andersen and Christiansen [43], Andersen et al. [44]. By other procedure, Mackenzie and Boyle [45], Ponter and Carter [46], Maier et al. [47], Boulbibane and Ponter [48] used the elastic compensation method considered as a direct method for nonlinear programming technique. In those studies, Young’s modulus of each element is modified during the iterative linear-elastic finite element, then the optimized statically admissible stress field is obtained after each iteration leading to an upper bound and a pseudo-lower bound solution. Similarly to the linearizing technique, the high expense of computation is the major obstacle of this procedure.

Recently, a state of art primal-dual interior point algorithm has been introduced, the nonlinear conditions of the yield functions can be transformed into the form of the second order cone programming (SOCP) problem with a large number of variables and nonlinear constraints. Then the solution of a minimization problem with linear objective function and feasible region defined by some cones. The advantage of this method is the ability to solve large problems with thousands of variables in tens of seconds only. The important contributions to this method can be seen in studies of Nesterov et al. [49], Andersen et al. [50], Ben-Tal and Nemirovski [51], Renegar [52], Makrodimopoulos and Bisbos [53], Bisbos et al. [54], Makrodimopoulos [55].

1.2.3 Discretization techniques

Theorems of limit and shakedown analysis lead to two classic problems including static and kinematic formulations corresponding to the lower-bound and upper-bound problems, respectively. The lower-bound solution will be obtained using equilibrium formulation, and the stress or moment fields associated with the nodal values are discretized. The approximated fields must satisfy the boundary conditions,
Chapter 1. Introduction

the equilibrium conditions and the fulfill of yield criterion. In order to satisfy this
statically admissible conditions, a set of linear constrains on the stress or moment
parameters needs to be introduced. Therefore, approximating the stress field is more
difficult than those of displacement or velocity fields. The displacement or velocity
formulation requires an approximation of a kinematically admissible displacement
velocity field, and the upper-bound solution will be obtained. The internal com-
patibility condition can be straightforwardly satisfied in the assembly scheme, and
the boundary conditions can be enforced directly. A number of studies based on
numerical method, such as finite element method (FEM), smoothed finite element
method (SFEM), or mesh-free methods were carried out for limit and shakedown
problems.

Nowadays, finite element method has become the most popular tool in academic
as well as industrial applications. In the literature, there are three basic types of
finite element models, i.e., displacement, equilibrium and mixed formulations. In
case of limit analysis, equilibrium model has been investigated in studies of Hodge
and Belytschko [56], Nguyen [57], Krabbenhof and Damkilde [58], Lyamin and
Sloan [59], Le et al. [60]. Displacement finite element models can be found in works
of Hodge and Belytschko [56], Le et al. [60], Anderheggen [61], Krabbenhoft et al.
[62], Capsoni and Corradi [63], Bleyer and Buhan [64]. The mixed formulation allows
both stresses and displacements to be determined directly, and volumetric locking
can be avoided, but there is one drawback exiting, that is the solution obtained is
lack of information on the status, it is unclear whether the solution is upper bound
or lower bound. Mixed approach for limit analysis was developed by Christiansen
[39], Capsoni [65], Yu and Tin-Loi [66]. Finite shakedown formulation combined
with different optimization algorithms, e.g., piecewise-linear yield criteria, Newton-
type scheme or interior-point method were developed. The contribution of this field
can be seen in works of Belytschko [67], Tin-Loi [38], Carvelli et al. [68], Heitzer et
al. [21], Yan and Nguyen [69], Vu et al. [70, 71], Simon [72], Simon and Weichert
[73, 76]. Recently, FEM in combination with second-order cone programming was
also applied to solve limit and shakedown analysis in works of Tran et al. [77], Le
et al. [32]. An improved form of standard FEM so-called SFEM has been extended
to direct analysis in studies of Le et al. [78, 79], Tran et al. [80], Nguyen-Xuan et
al. [81], Ho et al. [82]. Besides FEM and SFEM, an other mesh-based procedure
named Boundary Element method (BEM) has been successfully applied for limit
and shakedown analysis, the contribution can be found in works of Maier and
Polizzotto [83], Panzeca [84], Zhang et al. [85], Liu et al. [86, 87].

In recent years, taking advantage of computational efficiency, mesh-free methods have been continuously developed and significantly devoted to the development of limit and shakedown analysis. Natural Element method was employed to handle limit and shakedown problems, see e.g. Zhou et al. [88, 89]. Application of Element-free Galerkin method combined with the non-linear programming for solving optimization problems can be found in works of Chen et al. [90, 91]. Le et al. [92–95] also adopted EFG method by combining with stabilized conforming nodal integration (SCNI) and SOCP, then employed to solve upper bound as well as lower bound limit analysis. Similarly, the meshless based radial basis function so-called Radial Point Interpolation method was also using to deal with the upper-bound limit analysis problems, see e.g. Liu and Zhao [96].

1.2.4 The direct analysis for microstructures

The computation of heterogeneous microstructure were early carried out from 19th century by Voigt (1887) with the rule of mixtures. Then, several homogenization techniques, such as self-consistent [97], variational bounding methods [98, 99] and asymptotic homogenization [100, 101] have been proposed to handle the microstructures with assumptions of linear elastic behavior, simple geometries and small strains. Since the increasing use of composite materials and the requirement of dealing with the complex behavior of microstructures, a new class of so-called unit cell methods was early proposed by Eshelby [97] and widely applied in this field [102, 103]. This approach can provide the effective properties of the material as well as the valuable information on the local micro-structural fields. However, the unit cell methods are based on a priori assumed macroscopic constitutive relations, which is usually infeasible when the constitutive behaviour becomes non-linear. Therefore, most of above techniques are unable in large deformations, complex loading paths or the change of geometries. In recent years, the multi-scale homogenization technique or also called global-local analysis firstly proposed by Suquest [104] has been widely exploited. The computational homogenization methodology have been mostly applied to the periodic composite and heterogeneous materials. Techniques of computational homogenization can overcome the major drawbacks of unit cell methods, provide transition between micro-scale features and macro-response, and allow the use of modelling technique on microscopic structures as finite element
method \[105\)–\[107\], the Voronoi cell method \[108\)–\[109\], a crystal plasticity framework \[110\], boundary element method \[111\], mesh-free methods \[112\)–\[113\].

Extending to predict the macroscopic behavior of composite materials, Suquet \[114\] introduced the homogenization theory to plastic mechanics. Based on the concept of RVE and homogenization technique of Suquet, Buhan and Taliercio \[115\] proposed the first formulation of limit analysis in terms of solving the composite structure at micro-scale. The theoretical formulation then developed in the studies of Taliercio \[116\], Taliercio and Sagramoso \[117\] for fiber-reinforced composite using Drucker-Prager, Mohr-Coulomb or von Mises yield criterion. The first numerical implementation for this field belongs to Francescato and Pastor \[118\] with the use of finite element method and linear mathematical programming. By means of static direct methods, Weichert et al. \[25\)–\[119\] developed a 3-dimensions finite element procedure for analysis of isotropic microstructures. Using a similar approach, Zhang et al. \[120\] presented the quasi-lower bound formulation for periodic composite and heterogeneous materials using the nonlinear programming. Besides that, the kinematic formulations in combination with nonlinear algorithms can be found in studies of Li et al. \[121\)–\[125\]. In these works, both isotropic and anisotropic materials obeying the von Mises or elliptic yield criterion were considered. For the purpose of improving the computational aspect, Le et al. \[126\] proposed a numerical method based on the finite element method and the combination of kinematic theorem and homogenization theory for limit analysis of periodic composite. The study proved that the accurate solutions can be obtained rapidly using SOCP.

1.2.5 Mesh-free methods - state of the art

The necessary of mesh-free methods

Parallel with the development of information technology and computer, the numerical methods become indispensable tools for simulation and design of practical structures. The engineering problems are usually formulated in form of Partial differential equations (PDEs) relating to the boundary conditions, and solved using analytical method. The complex problems need to be approximated using the numerical methods, for which the PDEs are transferred to an equilibrium form so-called variational form or weak form, then a set of simultaneous algebraic equations is established for overall computational domain via the approximate functions. The
boundary conditions are required to be applied before solving the problem to determine the approximate solutions.

As mentioned, among numerical schemes, FEM have been rapidly developed and became the most popular tools in simulation as well as analysis of engineering problems. Various codes or commercial software packages based on FEs background are developed and widely used in almost areas, for example structural mechanics, thermal analysis, fluid dynamics, and even multi-physics simulation. In FEM, the computational domain is sub-divided to various finite elements connected together at nodes. This work is called discretization; and the nodal connectivity well-known as the mesh is the fundamental feature of mesh-based method. The creation of the mesh plays an important role in FEM implementation and takes most of total computational cost. There are several issues generated by the mesh, for example in large deformation problems, the continuous remeshing of domain may be required to avoid the breakdown of the computation caused by the excessive mesh distortion. The very fine mesh may be required for the accurate solutions, that makes the computational cost increase. In one other case, fracture problems, FEM may fail in dealing with the discontinuities at crack paths and crack tips where the refinement is required after every computational step. Therefore, no-mesh is necessary in whole process of solving problems, and that is the ideal for a novel scheme named mesh-free or meshless method.

Generally, dealing with engineering problems, the numerical implementation of mesh-free (MF) methods is similar to mesh-based ones, see Figure 1.4. The major difference between MF scheme and FEM is the strategy to discretize the computational domain and construct the shape function. The nodal connectivity is not required in mesh-free methods (Figure 1.2). The absence of mesh is the most attractive characteristic of MF methods leading to the reduction of computational cost [127] and the flexibility in operation of nodes (adding, eliminating or moving nodes) within the computational domain. Owing to that advantage, the adaptive technique as $p$—adaptive or $h$—adaptive can be conveniently applied in MF method [127]. The computation is also flexibly implemented using several procedures. Some mesh-free models use Gauss points relating to background cells as Figure 1.3(a), that is similar to FEM and does not ensure the truly meshless feature. In other methods, the Gauss points are replaced by the scattered nodes within the problem domain. Then, the nodal computational domain (or representative domain) can be determined using various different means, for example Voronoi diagram known as
the duality of Delaunay triangles as Figure 1.3(a). For convenience, the available Voronoi function in programming language software, e.g. Matlab, C++ or Python, can be utilized.

![FEM discretization](image1)

![MF discretization](image2)

Figure 1.2: The discretization of FEM and MF method

![Based on Gauss points](image3)

![Based on scattered nodes](image4)

Figure 1.3: The computational domain in mesh-free method

An other difference of mesh-free methods compared with mesh-based procedures is the influent domain (or support domain). The concept *influent domain* is used in case of the computation is the carried out on scattered nodes, whereas the concept *support domain* is used when the implementation bases on arbitrary point within the computational domain, e.g. Gauss point. For convenience, the concept *influent domain* used in the thesis. While the nodal influent domains in mesh-based methods
are limited by all elements attached to nodes, in meshless method, the influent domains of nodes can be flexibly chosen (rectangle domain, square domain or circular domain). That domains can overlap as seen in Figure 1.2(b), and can be resized easily. That ensures the good continuity for MF approximation in comparison with the traditional approaches. Figure 1.2(b) illustrates the strategy to determined the influent domain using the circle where the influent radius $R_I$ can be modified via a dimensionless parameter $\beta_s$ as

$$R_I = \beta_s d_I, \quad \beta_s \geq 1$$

where $d_I$ denotes the minimal distance from considered node to its neighbours in the computational domain. The accuracy and computational expensive depend on the choice of influent radius. Therefore, parameter $R_I$ needs to be investigated in the numerical implementation.

The most important advantages of mesh-free methods in comparison with FEM is the high-order continuous shape function. As a consequence, the MF methods can provide highly accurate solutions with the good convergence rate [128]. Moreover, the accuracy of solutions in MF method can be easily improved via the modification of influent domain. The most common drawbacks of MF methods are probably the computational cost when constructing the shape function, the density of matrices and the lack of Kronecker-delta property in several approximation techniques.

Recently, various modes of meshless method have been developed, improved and widely applied in different areas, such as solid mechanics, fluid mechanics, molecular dynamics or even molecular biology. Each method bases on an individual basis function and uses an individual approximation or interpolation technique, more details will be presented in the following sections.

**Overview of popular mesh-free methods**

The original mesh-free method is Smooth Particle Hydrodynamics (SPH) introduced by Gingold and Monaghan [129] and Lucy [130]. SPH method firstly applied to simulate the phenomena such as supernova, and then was employed in fluid dynamics. Libersky and Petschek [131] extended this method to solid mechanics analysis. The main advantage of the SPH method is its ability to treat local deformations, which is considered to be better than mesh-based methods. Then, this
Figure 1.4: Numerical procedures: Mesh-free method versus FEM
feature was utilized to handle a number of problems as metal form or crack propagation analysis, etc. However, the classical formulations of SPH lack of stability and consistency, thus various modifications have been carried out to improve the accuracy in recent years.

Based on the ideal of SPH method, Belytschko et al. [132] developed the Element-free Galerkin (EFG) method using Moving Least Square (MLS) approximation priorly proposed by Lancaster and Salkauskas (1981) [133]. EFG method avoids the discontinuous feature of previous SPH versions and becomes the most widely used meshless method. Liu et al. [134] proposed the meshless method named Reproducing Kernel Particle Method (RKPM). Although the method is similar to EFG procedure, RKMP originally bases on the wavelets rather than on curve-fitting. Surprisingly, the polynomial reproduction leads to the shape function of RKPM almost identical to one of EFG scheme. At the same time, Duarte and Oden [135] introduced hp-cloud method. That is the first mesh-free method developed without relying on the idea of SPH. In contrast to the EFG and RKPM method, hp-cloud method uses an extrinsic basis to increase the consistency (or completeness) of expression. Based on the similarities between meshless method and finite element those, Melenk et al. [136] formulated a consolidated form of them so-called Partition of Unity Finite Element Method (PUFEM). One of popular mesh-free procedures named Meshless Local Petrov-Galerkin (MLPG) was developed by Atluri et al. [137]. The difference of MLPG compared to above mentioned methods is the local weak form constructed on overlap sub-domains alternating to the global weak form, and then the integrals are calculated in that local domains. Using MLPG approach, the integration can be implemented without the background cells, ensuring that MLPG is a truly meshless method. Arroyo and Ortiz [138] proposed a mesh-free method based on the Local maximum-entropy approximation (LMEA). The basis function used in this method is similar to one in MLS, but its advantage is that the local approximation function produces a shape function nearly satisfying Kronecker-delta property at the boundaries of problem domain. Using Natural Neighbour Interpolation (NNI) technique introduced by Sibson [139], Brauand Sambridge [140] developed the Natural Element method (NEM) for the purpose of solving PDEs. NEM was then extended to solid mechanics analysis by Sukumar et al. [141].

Besides, several meshless methods were developed based on the interpolation technique, and the radial basis functions (RBFs) are commonly utilized. The fun-
damentals of RBF method were firstly introduced by Hardy \cite{142} for cartography problem. In this study, the multiquadric (MQ) radial function was presented. Lately, Franke \cite{143} investigated 32 most commonly used interpolation methods and proved that MQ is the best one. The main feature of MQ method is the basis function only depends on the Euclidean distance which is radially symmetric to its center. From MQ method, different radial functions was generalized as the thin plate spline, the Gaussian, the cubic, etc, constituting the so-called Radial basis function method, see e.g. Duchon \cite{144}. Recently, Kansa \cite{145,146} introduced RBF collocation method considered as a way to solve the partial differential equations (PDEs) for parabolic, elliptic and hyperbolic. Kansa’s method bases on the collocation method and the MQ-RBF, yielding to the global approximation. In this scheme, the dense stiffness matrix is obtained. As a result, it takes a very expensive cost to solve the problems with a large number of collocation points. Avoiding that drawback, the RBF Hermite-Collocation was proposed, both globally and locally supported RBF was used. The results proved that globally supported RBF gives the more accurate solution than locally supported case, but its computational expense is higher. Recently, the RBF method named Local Multiquadric was proposed by Lee et al. \cite{147}, for which the approximate function is constructed using sub-domains, then the local approximation and a sparse stiffness matrix are obtained. Applying weak form, Wendland \cite{148} developed a Galerkin mesh-free method using radial basis functions. RBFs have also been used in Boundary element method (BEM) or Meshless local Petrov-Galerkin (MLPG) and successfully applied to solve various nonlinear problems in computational mechanics.

With the purpose of handling the matters generated by the lack of Kronecker-delta property in mesh-free approximations, Liu and Gu developed the Point Interpolation Method (PIM) using the polynomial basis function. PIM encounters drawback in inverting matrices vanished in some situations, thus an alternative one so-called Radial Point Interpolation Method (RPIM) was introduced \cite{149}. The major advantage of using radial basis function in PIM is the invertibility of moment matrix. Unfortunately, the accuracy of results may not be given as expected. As a result, a polynomial term is added into the basis function to improve the accuracy as well as the stability of solutions. Using the radial basis functions but approaching in the opposite direction compared with RPIM, Tran-Cong et al. \cite{150,153} developed the Integrated radial basis function (iRBF) method. Generally, iRBF method possesses all the good features of RPIM, but its approximation is better than RPIM.
one owing to the use of multiple integration, yielding to the higher-order shape function.

**Approximation technique based on RBFs**

The key ideal of mesh-free methods is that the approximation or interpolation bases on a set of arbitrary scattered nodes. To ensure the convergence and stability, the approximate functions must satisfy following requirements.

- **Consistency:** If $s$ is the order of the highest derivative occurring in the weak form, the approximate function should be differentiable at least up to the order $s^{th}$ inside influent domain.

- **Completeness:** The shape function must have ability to reproduce polynomials up to order $s^{th}$ to ensure the stability and convergence of numerical method. If the shape function $\Phi_I(x)$ is complete up to order $s^{th}$, any degree $s^{th}$ polynomial can be reproduced as

$$\sum_I x^p \Phi_I(x) = x^p, \quad 0 \leq p \leq s \quad (1.2)$$

or

$$\sum_I p(x) \Phi_I(x) = p(x), \quad \forall x \in \Omega \quad (1.3)$$

where $p(x)$ denotes the basis function.

- **Partitions of unity:** The sum of all nodal shape function values at any point in the computational domain must be unit, ensuring the proper representation of a constant field of the solid

$$\sum_I \Phi_I(x) = 1, \quad \text{in } \Omega_I \quad (1.4)$$

As mentioned, up to now, mesh-free method have been considerably developed with a number of schemes based on different approximation or interpolation techniques such as SPH method [129, 130], RKPM method [134], PU method [136, 154, 155], RBFs method [145, 153], MLS approximation [132], LMEA approximation [138], NNI approximation [139, 141], and various other approaches, more details can be found in [156–158]. Owing to its high-order shape function and the
advantage in enforcing essential boundary conditions, this thesis only focuses on the iRBF method proposed by Tran-Cong et al. [150–152], where the radial basis functions are used.

Several popular radial basis functions in the literate can be listed as follows

\[
g_I(x) = \begin{cases} 
(r^2 + a_I^2)^q & \text{for general multiquadrics} \\
\sqrt{r^2 + a_I^2} & \text{for multiquadrics (MQ)} \\
\frac{1}{\sqrt{r^2 + a_I^2}} & \text{for inverse multiquadrics} \\
\frac{1}{r^2 + a_I^2} & \text{for inverse quadrics} \\
e^{-\left(\frac{r}{a_I}\right)^2} & \text{for Gaussian} \\
r^2 \log(r) & \text{for thin plate spline}
\end{cases}
\]

(1.5)

where \( r = \|x - x_I\| \) is the radius from node \( I^{th} \) and others in the influent domain; the shape parameter \( a_I = \alpha_s d_I \), with \( \alpha_s > 0 \) is the dimensionless factor; and \( d_I \) denotes the minimum distance from node \( I^{th} \) to its neighbours.

**Direct (dRBF) and indirect/integrated (iRBF) formulations**

A smooth function \( u(x) \) can be directly approximated based on a set of \( N \) scattered nodes and a radial basis function as

\[
uh(x) = \sum_{I=1}^{N} g_I(x)a_I
\]

(1.6)

where \( uh(x) \) is the approximate function of \( u(x) \); \( \{a_I\}_{I=1}^{N} \) is a set of expanded (or unknown) parameter; \( \{g_I(x)\}_{I=1}^{N} \) is the radial basis function.

From (1.6), the derivatives of \( u(x) \) can be calculated as

\[
uh_{j...l}(x) = \frac{\partial^k uh}{\partial x_j \ldots \partial x_l} = \sum_{i=1}^{N} \frac{\partial^k g_I(x)}{\partial x_j \ldots \partial x_l} u_I
\]

(1.7)

It is worth mentioning that errors in the approximate function computed by
Equation (1.6) is low, but errors in its derivatives are still high [152]. Moreover, the derivative functions, especially higher order ones, are strongly influenced by the local behavior of the approximation. Consequently, the so-called indirect RBF method was also developed in [150–153] and will be recalled the following.

In iRBF approach, the highest derivative (order $s^{th}$) of approximate function is firstly constructed using RBF as

$$u_{jkr\ldots rs}^h(x) = \sum_{I=1}^{N} g_I(x)a_I \tag{1.8}$$

Next, the lower-order derivatives and the original function will be calculated using the multiple integration as follows

$$u_{jkr\ldots r}^h(x) = \int \sum_{I=1}^{N} g_I(x)a_I dx_s + C_s(x) \tag{1.9a}$$

$$\ldots$$

$$u_{j}^h(x) = \int \cdots \int \sum_{I=1}^{N} g_I(x)a_I dx_s \cdots dx_k + C_{s\ldots k}(x) \tag{1.9b}$$

$$u^h(x) = \int \cdots \int \sum_{I=1}^{N} g_I(x)a_I dx_s \cdots dx_j + C_{s\ldots j}(x) \tag{1.9c}$$

where $C_s(x), \ldots, C_{s\ldots j}(x)$ is the function order 0 up to order $(s-1)^{th}$ with the parameters are integral constant.

Playing the prior inversion of matrix in (1.9c) and substituting to Equations (1.8) and (1.9a - 1.9c), the reflection of the approximate function and its derivatives pass through the nodal values can be obtained. It should be noted that the iRBF shape function satisfies Kronecker-delta property leading to the essential boundary conditions can be applied similarly to the finite element method.

**Numerical implementation in mesh-free method**

The engineering problems are firstly formulated in form of PDEs with the boundary conditions, and then solved to obtain solutions. Consider a PDEs in domain $\Omega$ with kinematic boundary $\Gamma_u$ and static boundary $\Gamma_t$ such that $\Gamma_u \cup \Gamma_t = \Gamma$ and
Γ_u ∩ Γ_t = ∅ as follows

\begin{align*}
\nabla_s^T \sigma + b &= 0, \quad \text{in } \Omega \quad (1.10a) \\
\sigma &= D \nabla_s u \quad (1.10b)
\end{align*}

where \( \sigma \) is the Cauchy stress tensor; \( b \) is the body force per a volume unit; \( \nabla \) is the differential operation; \( D \) is the matrix consisting material constants; \( u \) is the vector including displacement components. The PDEs \((1.10a)\) can be rewritten as

\[ \nabla_s^T D \nabla_s u + b = 0 \quad (1.11) \]

The boundary conditions are defined by

\begin{align*}
\mathbf{n} \cdot \sigma &= \bar{t}, \quad \text{on } \Gamma_t \quad (1.12a) \\
u &= \bar{u}, \quad \text{on } \Gamma_u \quad (1.12b)
\end{align*}

with \( \mathbf{n} \) is the outward normal vector of static boundary.

The equation system \((1.11)\) is the strong form describing the mechanical behaviors where displacements is the main variables. Almost engineering problems will be solved using numerical procedures after transforming to PDEs form. There are two main strategies for solving problems, which are known as strong form and weak form, respectively, and will be clarified in the following.

**Strong form - Collocation method**

Consider an approximation for set of \( N \) discretized nodes as

\[ u^h(x) = \sum_{I=1}^{N} \Phi_I(x) u_I \quad (1.13) \]

with \( \Phi_I(x) \) is the shape function obtained using the approximate/interpolated techniques previous presented; \( u_I \) denotes the unknown values at nodes. For the strong form methods, the order of approximate functions must be higher or equal to the order of derivative of strong form equation system, and that requirement is called \textit{strong}.

In collocation method, the equation system \((1.11)\) is satisfied at every points in
the problem domain, and the conditions in \((1.12)\) are applied

\[
\nabla^T_s D \nabla_s u(x_J) + b_J = 0, \quad \forall J \in \Omega \tag{1.14a}
\]

\[
n D \nabla_s u(x_K) = \bar{t}_K, \quad \forall K \in \Gamma_t \tag{1.14b}
\]

\[
u(x_L) = \bar{u}_L, \quad \forall L \in \Gamma_u, \quad (J + K + L = N) \tag{1.14c}
\]

The advantage of collocation approaches is the simple implementation and computational speed. There are no need of integrals, and the shape conditions are directly enforced at nodes instead of at Gauss points. However, unexpected solutions can be obtained due to the instability. In this thesis, the so-called Stability conforming nodal integration (SCNI) scheme will be employed to handle this drawback. Using SCNI technique, problems can be implemented by the similar way of collocation method, but the stability and accuracy of solutions are ensured.

**Galerkin weak form**

The unknown field will be approximated via a trial function \(u\). Multiplying both sides of strong form \((1.11)\) with an arbitrary trial function \(\varphi\) and carrying out the integration on overall domain \(\Omega\), the weak form will be obtained as

\[
\int_{\Omega} \varphi^T \nabla^T_s D \nabla_s u d\Omega + \int_{\Omega} \varphi^T b d\Omega = 0 \tag{1.15}
\]

Using the partial integral, then applying the static boundary condition \((1.12a)\) and the condition \(\varphi = 0\) on \(\Gamma_u\), the weak form \((1.15)\) can be rewritten as

\[
\int_{\Omega} (\nabla_s \varphi)^T D (\nabla_s u) d\Omega = \int_{\Omega} \varphi^T b d\Omega + \int_{\Gamma_t} \varphi^T \bar{t} d\Gamma \tag{1.16}
\]

If in the strong form, the PDEs are required to be satisfied at every points in problem domain and the approximate function must have order at least equal to those in the highest derivative of PDEs, in the weak form, applying the partial integral for \((1.15)\) leads to the reduction of the order of operator \(\nabla\). Therefore, the requirement of continuity is *weak*, meaning that the order of trial function can be smaller than the order of highest derivative in PDEs and all conditions need to be satisfied only inside domain \(\Omega\). Noting that when transforming from strong form to weak form, the static condition \((1.12a)\) is used, thus there is only the displacement
condition (1.12b) in the weak form.

Using weak form, the engineers usually prefer to directly access the Galerkin weak form

\[ \int_\Omega (\nabla s \delta u)^T D (\nabla s u) d\Omega - \left[ \int_\Omega u^T b d\Omega + \int_{\Gamma_t} \delta u^T t d\Gamma \right] = 0 \] (1.17)

with \( \delta \) is the variational operator. Considering physical meaning, a displacement field satisfying Equation (1.17) with the arbitrary test function \( \varphi \) will minimize the total power in the whole system and keep the system in the stable and equilibrium state. Equation (1.17) is completely similar to equation (1.16) constructed from the strong form.

The mesh-free method based on Galerkin formulation can be found in the studies of Belytschko et al. [132, 159, 160], Liu et al. [134], Duarte and Oden [135] or Melenk and Babuska [136]. Two major aspects of this method including applying the essential boundary condition and estimating the integrals in the weak form equations will be discussed in following sections.

**Enforcement of essential boundary conditions**

Usually in mechanics problems, when considering behavior at elastic state, after constructing the stiffness matrix, it is in need to eliminate the singularity caused by the physical movement of the body, this work is called enforcing the essential boundary conditions. In order to easily impose this conditions, the shape functions are required to satisfy Kronecker-delta property, it means

\[ \Phi_I(x_J) = \delta_{IJ} = \begin{cases} 1 & \text{if } I = J, \\ 0 & \text{if } I \neq J. \end{cases} \] (1.18)

While Kronecker-delta property obviously exists in FEM, most of meshless approaches lack this feature. Consequently, when applying the boundary conditions, several special techniques will be employed such as Lagrange multiplier [132, 134–136], penalty method [161, 162], modified variational principle [159], point collocation method [161], coupling to with FEM [160, 163] or specially modified shape function [164]. For the case of RPIM and iRBF approaches, the essential boundary conditions can be imposed similarly to the finite element method.
Numerical integration

In the numerical methods, the use of numerical integration is the essential work to evaluate the integrals on the computational domain. Mesh-free methods usually employ two main schemes, there are using Gauss integral based on the background cells and using nodal integral based on discretized nodes. Among them, Gauss integral is the most popular technique for numerical methods. The drawback of this scheme is the requirement of background cells, making the procedure not truly meshless. In order to obtain a good description of the high-order shape function, a number of Gauss points are required in the domain, increasing the computational cost. Moreover, if the background cells are used in Galerkin weak form, the numerical integration errors (with Gauss quadrature) occurs in all mesh-free approximation owing to the support domains for the basis functions do not coincide with the background cells. Consequently, instead of using Gauss integral, Beissel and Belytschko [165] proposed the modification of power functional by adding a square of residual weight to the equilibrium equation in order to eliminate the singularity. In other research, Chen et al. [166] introduced the Stability conforming nodal integration (SCNI) technique based on the idea of smoothing strain rate at node. Then, to improve this technique in terms of accuracy, stability and convergence rate, Chen et al. [167] proposed to add a reinforced linear function into the approximation. The scheme was used in combination with Moving Reproducing Kernel Particle Method (MRKPM) in [167].

1.3 Research motivation

Numerical methods are the most efficient tools for current studies in the field of limit and shakedown analysis. As mentioned above, a number of researchers have devoted their effort to develop the robust approaches for this area. The numerical procedures using continuous field, semi-continuous field (Krabbenhoft et al. [62]), or truly discontinuous field (Smith and Gilbert [168]) have been executed with the support of finite element method (FEM). However, there are several matters of mesh-based procedures, which need to be handled, for instance, locking problems, mesh distortion and highly sensitive to the geometry of the original mesh, particularly in the region of stress or displacement singularities. In order to improve the computational aspect of FEM, a number of studies proposed the adaptive tech-
nique for limit and shakedown analysis, the achievement can be found in the works of Christiansen and Pedersen [169], Borges et al. [170], Franco et al. [171], Lyamin and Sloan [172], Cecot [173], Ngo and Tin-Loi [174], Ciria et al. [175], Le [176]. However, the whole process is complicate and requires the fine meshing to obtain the expected results. An improving form of FEM named SFEM (smoothed finite element method) is also applied in works of Le et al. [78, 79], Tran et al. [80], Nguyen-Xuan et al. [81]. Generally, SFEM is better than FEM in terms of stability and convergence, but this method does not surmount all disadvantages of FEM caused by the mesh. Recently, mesh-free methods are also extended to direct analysis. Among them, Element-free Galerkin (EFG) method is the most interested choice, several typical studies can be noted here as Chen et al. [90, 91], Le et al. [92, 95]. Besides, some other meshless procedures have been also successfully applied to this area such as Natural Element method (NEM - Zhou et al. [88, 89]), Radial Point Interpolation method (RPIM - Liu and Zhao [96]). In comparison with the traditional approaches, mesh-free methods possess the high-order shape function, hence above disadvantages can be overcame. However, it should be noted that several meshless methods lack Kronecker-delta property leading to the difficulty in imposing the essential boundary conditions. Owing to the advantages of shape function as mentioned in previous sections, iRBF method can provide an efficient treatment for those obstacles arising in whole process of formulating and solving optimization problems. According to the author’s knowledge, the applications of iRBF method are focused on the fields of solving PDEs [150–153], fluid mechanics [177], or elastic analysis of solid and fracture mechanics [178]. The development of iRBF method for limit and shakedown analysis will be a new contribution to this area. In addition, in previous studies using iRBF, the numerical integration is carried out utilizing Gauss points, increasing the computational cost. Therefore, the stabilized approximation based on the combination of iRBF approximation and SCNI will improve the computational aspect of proposed numerical method.

Moreover, solving limit and shakedown problem requires to handle the optimization problem involving either linear or non-linear constrains. The traditional way to overcome this drawback is linearizing non-linear convex yield criteria. The efficient tools, for instance, Simplex algorithm (Anderheggen and Knopfel [33], Christiansen [179]), can be used. However, a large number of constrains and variables in the optimization problems are required to obtain the sufficiently accuracy results, which increase the computational cost. On one other hand, that is the attempts to deal
with the convex yield criteria using non-linear packages. Although the highly accurate solutions can be obtained, the expensive cost is the major trouble of this scheme. In framework of limit analysis, the primal-dual interior-point algorithm (Christiansen and Kortanek [180], Andersen and Christiansen [181]) is well-known as one of most robust and efficient algorithms in handling the optimization problems with large-scale nonlinear constrains. Therefore, extending of this scheme to the shakedown formulation will lead to more advantages for direct analysis of either structures or materials.

Besides, the earliest application of direct analysis for microscopic structures can be found in studies of Buhan and Taliercio [115], Taliercio [116], Taliercio and Sagramoso [117], where the limit load of typical problems were determined. The homogenization theory was applied to limit analysis using linear programming in works of Francescato and Pastor [118], Zhang et al. [120], Weichert et al. [25, 119], Chen et al. [182]. Besides that, the nonlinear programming were also employed for direct analysis of heterogeneous materials by Carvelli et al. [183], Li et al. [121–125], Hachemi et al. [184], Le et al. [126]. Actually, almost studies dealt with the isotropic or anisotropic materials using linear or nonlinear programming with the support of finite element method, the application of mesh-free method in framework of computational homogenization analysis of materials at limit state is still unavailable.

In conclusion, it can be observed that many challenges still remain in developing a robust tool to improve the computational aspect of limit and shakedown analysis for structures and materials. The absence of the integration of an optimization algorithm in structural analysis software packages, e.g. ANSYS or ABAQUS, leads to the fact that limit and shakedown analysis has not yet been commercialized and widely applied in structural design. Present study focuses on the combination of a discretization scheme and an optimization programming to propose an efficient numerical approach for direct analysis method, i.e., (i) the stabilized iRBF mesh-free method will be developed; (ii) the optimization problems will be formulated using the so-called second-order cone programming (SOCP) to deal with the convex yield criterion; (iii) the numerical approach will be applied to handle the direct analysis problems for structures and materials.
1.4 The objectives and scope of thesis

The major objective of thesis is developing the integrated radial basis functions-based mesh-free method (iRBF method) and the optimization algorithm based on conic programming, then extending the numerical approach to limit and shakedown analysis of structures and materials. In order to obtain above mentioned aims, the following tasks will be carried out.

First, the mesh-free method based on integrated radial basis functions, for which the stability conforming nodal integration (SCNI) is employed to obtained the smoothed versions of shape function derivatives, is developed to discretize the computational domain; then the general approximate fields for different types of problems (displacement and stress fields) are established.

Second, the kinematic and static formulations of limit and shakedown analysis for structures and materials, governed by several well-known yield criterion, e.g., von Mises or Nielsen, are formulated, and then the optimization problems are cast as second-order cone programming.

Finally, the resulting optimization problems are solved using the highly efficient tools such as Mosek software package combined with Matlab programming. The obtained solutions are compared with other those in available studies in order to estimate the computational aspect of proposed approaches.

It is important to note that, within the scope of the thesis, proposed numerical method will be employed to deal with several common engineering structures, such as continuous beam, simple frame, plates, reinforced concrete slabs, or computational homogenization analysis of micro-structures. The material model is assumed as rigid-perfectly plastic or elastic-perfectly plastic. The 2D and 3D structures are considered under both constant and variable loads, corresponding to limit and shakedown analysis, respectively. The benchmark problems will be investigated for the comparison purpose; thereby, the computational aspect of proposed approach is evaluated.

1.5 Original contributions of the thesis

According to the author’s knowledge, the following points have never been published in other studies, and they can be considered as the original contributions of
Chapter 1

Introduction

• The development of stabilized iRBF method, which is based on the combination of iRBF approximation and SCNI scheme, for the field of limit and shakedown analysis.

• The development of stabilized iRBF method for computational homogenization analysis of micro-structures at limit state. This is the first time a mesh-free method is employed to treat that problem.

• Based on iRBF approximation and bounding theorems, the kinematic and static limit and shakedown analysis are formulated in form of SOCP. Proposed method is used to deal with various types of structures and materials obeying different yield criterion.

1.6 Thesis outline

The thesis includes 7 chapters, in which chapters 3, 4, 5 and 6 present the contents and numerical solutions collected from the manuscripts published or submitted to publication. The outline of the thesis is the following.

Chapter 1 generally introduces the thesis; the review of contents relating to the thesis, summarizes the historical development, applications and the contributions of available numerical procedures for engineering problems. Besides, the research motivations, objectives and scope of the thesis are also clarified.

Chapter 2 expresses the fundamental theories applied in the thesis involving limit and shakedown analysis, the optimization algorithms (second-order cone programming), the homogenization theory and the integrated radial basis function-based mesh-free method.

Chapter 3 presents the application of iRBF method for limit analysis of plane problem where both kinematic and static field corresponding to the upper and lower bound formulations are investigated. The obtained limit load multipliers for various benchmark problems are compared with those reported in previous studies.

Chapter 4 expresses the other application of iRBF method for limit analysis of reinforced concrete slabs under bending loads. The kinematic formulation of problems are considered, then the limit load and the collapse mechanism of slabs with
various different shapes are determined. The computational efficiency of proposed
method is evaluated via the comparison with other studies.

Chapter 5 presents the stabilized iRBF method in the application for limit and
shakedown analysis of plane problems using either two dimensional or three di-

dimensional models. The equilibrium formulation is employed in this section. The
quasi-lower bound of limit and shakedown loads are obtained owing to several spe-
cial techniques. The elastic stress field, residual stress field as well as the plastic
stress field for various different problems are illustrated. As previous chapters, the
computational aspect is also analysis.

Chapter 6 investigates the computational homogenization analysis of materials
using stabilized iRBF method. The approximate results involving limit load multi-
piers and the yield surface of materials are expressed.

Chapter 7 presents the discussions on the numerical solutions, the convergence
and reliability of results obtained using proposed method. Finally, several conclu-
sions are drawn and the recommendation for future works are also presented.
Chapter 2

Fundamentals

Previous chapter presents the literature review of limit and shakedown theory, homogenization technique as well as different approximation schemes of mesh-free methods. This chapter will clarify fundamentals applied to this thesis. The iRBF based meshless method proposed by Tran-Cong et al. [150–153] will be used to handle all problems in the thesis.

2.1 Plasticity relations in direct analysis

2.1.1 Material models

In plasticity theory, for convenience, the real behaviour of materials are replaced by the idealized models for which the hardening or softening behaviour can be ignored. Figure 2.1(a) illustrates the elastic-perfectly plastic model where material behavior is considered in two stages including elasticity and plasticity. In this model, materials behave elastically when stress is below the ultimate strength; otherwise, the yield occurs. In fact, the elastic deformations are very small in comparison with plastic ones; therefore, it can be ignored. In other words, the elastic-perfectly plastic model can be replaced by the rigid-perfectly plastic one as seen in Figure 2.1(b).

The plastic deformations obey the flow rule as

\[
\dot{\epsilon} = \mu \frac{\partial \psi}{\partial \sigma}
\]

where \(\mu > 0\) is plastic parameter; \(\psi(\sigma)\) denotes yield function forming the space limited by a time-independent yield surface such that

- \(\psi(\sigma) < 0\): elastic behavior;
Chapter 2. Fundamentals

(a) Elastic-perfectly plastic model  (b) Rigid-perfectly plastic model

Figure 2.1: Material models

- $\psi(\sigma) = 0$: appearance of plastic deformations;
- $\psi(\sigma) > 0$: inaccessible region.

The material models are required to obey the stability postulate proposed by Drucker and its important consequences called the normality rule and convexity.

**Drucker’s stability postulate**

Following Drucker, material models are stable if the work produced over the cycle of applied and removed loads is non-negative

$$\int (\sigma - \sigma^0) d\epsilon \geq 0$$  \hspace{1cm} (2.2)

where $\sigma$ is the current stress tensor on the yield surface ($\psi(\sigma) = 0$); $\sigma^0$ denotes the plasticly admissible stress tensor ($\psi(\sigma^0) < 0$).

Above formulae is also simply known as Drucker’s inequality and it is appropriate
for perfectly plastic and hardening materials

\[(\sigma - \sigma^0)\epsilon \geq 0\]  \hspace{1cm} (2.3)

Figure 2.2 describes the stable behavior (producing positive work) and unstable behavior (producing negative work). Following that, materials satisfy the behavior illustrated in Figure 2.2(a) will be called stable or standard materials; and the materials possess behavior according to those in Figures 2.2(b) and 2.2(c) will be called unstable or nonstandard materials.

The normality rule

\[\dot{\epsilon}_p \cdot \sigma = \sigma^0\]

\[\psi(\sigma) < 0\]

\[\psi(\sigma) = 0\]

Figure 2.3: The normality rule

The plastic strain rates are proportional to the gradient of yield functions at any point on the smoothed yield surface \(\psi(\sigma) = 0\), and are normal of yield surface. If there are singular points on the yield surface where the normal direction is not unique, the plastic strain vector must lie between adjacent normal at the corners as Figure 2.3. When \(n\) yield surfaces intersect at a singular point, formulae (2.1) is replaced by

\[\dot{\epsilon} = \sum_{i=1}^{n} \dot{\mu} \frac{\partial \psi}{\partial \sigma}\]  \hspace{1cm} (2.4)
The convexity

From Figure 2.3, it can be seen that corresponding to any stress $\sigma^0$ lies on the outward side of the tangent, Drucker’s inequality will be violated. In other word, if all of elastic stresses lie on one side of the tangent, the yield surface is convex. The material obeying to Drucker’s postulate is required that its yield function $\psi(\sigma)$ must be convex in the stress space $\sigma$. The convexity plays an important role in plasticity theory allowing the use of convex programming in direct analysis.

Yield criterion

A yield criterion defines the limit of elasticity under the complex stress state. For isotropic materials, the direction of principle stresses are independent with type of materials, thus the yield criterion can be preformed in terms of principle stresses as

$$\psi(\sigma_1, \sigma_2, \sigma_3) = k \quad (2.5)$$

where $k$ denotes the material constant, for example $k = \sigma_p$ in case of unaxial tension loading, or $k = \tau_p$ for unaxial shear loading.

For perfectly plastic materials, the yield function is independent with plastic strain rate, and due to the physical isotropic characteristic, it depends on the invariant of stress tensor. The yield criterion can be rewritten as

$$\psi(I_1, J_2, J_3) = k \quad (2.6)$$

where $I_1$ is the first stress invariant; $J_2$ and $J_3$ are the second and third invariant of deviatoric stress tensor.

In various problems, the results from experience demonstrate that for several materials, e.g. metal, the influence of hydrostatic stress is negligible. As a result, the yield function depends on the deviatoric stress tensor only, it means

$$\psi(J_2, J_3) - k_v = 0 \quad (2.7)$$

where $k_v^2 = \frac{\sigma_p}{\sqrt{3}}$, with $\sigma_p$ is the yield stress obtained form the unaxial tension test.

There are various yield criterion have been proposed for amount of materials, e.g. the Tressca or von Mises for metal, Drucker-Prager or Nielsen for reinforced...
concrete materials, Mohr-Column for soil, etc.

### 2.1.2 Variational principles

Consider an elastic-perfectly plastic or rigid-perfectly plastic body bounded by volume \( V \) with kinematic boundary \( \Omega_u \) where displacement components are constrained (Dirichlet boundary) and static boundary \( \Omega_t \) where stress components are known (Neuman boundary) such that \( \Omega_u \cup \Omega_t = V, \Omega_u \cap \Omega_t = \emptyset \). The structure is subjected to the body force \( f \) and surface load \( t \) as Figure 2.4.

![Figure 2.4: The equilibrium body](image)

The basic concepts

A stress field \( \sigma \) is called *admissible static* if it satisfies the equilibrium condition and static boundary condition

\[
\begin{align*}
\nabla \sigma &= f \quad \text{in } V \\
\mathbf{n} \sigma &= t \quad \text{on } \Omega_t
\end{align*}
\] (2.8a) (2.8b)
where $\nabla$ is the differential operator.

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \tag{2.9}$$

and $\mathbf{n}$ represents the outward normal matrix of surface $\Omega_t$

$$\mathbf{n} = \begin{bmatrix} nx & 0 & 0 & ny & 0 & nz \\ 0 & ny & 0 & nx & nz & 0 \\ 0 & 0 & nz & 0 & ny & nx \end{bmatrix} \tag{2.10}$$

A stress field $\mathbf{\sigma}$ is called \textit{plastically admissible} if the yield condition is not violated at anywhere

$$\psi(\mathbf{\sigma}) \leq 0 \tag{2.11}$$

A strain velocity field $\dot{\mathbf{\epsilon}}$ is called \textit{kinematically admissible} if it satisfies the compatible condition and kinematic boundary condition

$$\dot{\mathbf{\epsilon}} = \nabla \dot{\mathbf{u}} \quad \text{in} \ V \tag{2.12a}$$
$$\dot{\mathbf{u}} = 0 \quad \text{on} \ \Omega_u \tag{2.12b}$$

A strain velocity field $\dot{\mathbf{\epsilon}}$ is called \textit{plastically admissible} if strain vector is the normal of the yield surface and the external work is positive

$$\dot{W}_E = \int_V \mathbf{f}\dot{\mathbf{u}}dV + \int_{\Omega_u} \mathbf{t}\dot{\mathbf{u}}d\Omega_u > 0 \tag{2.13}$$

\textbf{Markov’s principle}

Following Markov, among all kinematically admissible and incompressible strain fields, the actual strain field will minimize the functional

$$\psi(\dot{\mathbf{u}}) = \int_V D_p(\dot{\mathbf{\epsilon}})dV - \left[ \int_V \mathbf{f}\dot{\mathbf{u}}dV + \int_{\Omega_u} \mathbf{t}\dot{\mathbf{u}}d\Omega_u \right] \tag{2.14}$$

32
where $W_I$ and $W_E$ are the power of internal load and external load, respectively; $D_p(\dot{\epsilon})$ denotes the plastic dissipation function defined by

$$D_p(\dot{\epsilon}) = \max_{\sigma} \sigma \dot{\epsilon} \quad \text{such that} \quad \psi(\sigma) \leq 0 \quad (2.15)$$

By solving problem (2.15), the stress state $\sigma(\epsilon)$ corresponding to the strain field $\dot{\epsilon}$ obeyed the normality rule will be obtained. The plastic dissipation function is now rewritten as

$$D_p(\dot{\epsilon}) = \sigma(\epsilon) \dot{\epsilon} \quad (2.16)$$

Noting that function $D_p(\dot{\epsilon})$ depends on the materials and the yield criterion used.

Hill’s principle

Following Hill, among all statically and plastically admissible stress fields, the actual field will minimize the functional

$$\Pi(\sigma) = -\int_{\Omega_u} (n\sigma) \dot{u} d\Omega \quad (2.17)$$

It is interesting to note that the consequences of principles formulated by Markov and Hill are well-known as the upper bound and lower bound of direct analysis which will be recalled following.

2.2 Shakedown analysis

In practice, structures can be subjected to various different forms of mechanical or thermal loading, for example monotonic or proportional loads, repeat loads, or even arbitrarily varying loads. Therefore, the failure of structures can be caused by various reasons. Under different intensities of applied loads, several behaviors of the structures can be obtained as follow

1. The respond of structure is only perfectly elastic if the load intensities are significantly small (Figure 2.5(a)).

2. If intensities of load beyond the elastic limit but are not too high, the plastic deformation occurs, increases and stops after some cycles. The behavior of
structure becomes elastic again. That state is called shakedown or adaptation (Figure 2.5(b)).

3. Under the load which is higher than the elastic limit, plastic deformation occurs and develops but it is not stable, the total strain is too large and structure becomes unserviceable. The phenomenon is called incremental collapse (Figure 2.5(c)).

4. Another behavior is called alternating plasticity. The plastic deformation change sign after every loading cycle, so the total strain is kept in small value. The structure will fail after a number of cycles because of the low-cycle fatigue failure (Figure 2.5(d)).

5. The structure can be collapse at the first cycle of loading due to the intensity of applied load is higher than its instantaneous load-carrying capacity. This situation is called plastic collapse (Figure 2.5(e)).

![Diagram of different behaviors of structures under cycle load]

Figure 2.5: The different behaviors of structures under the cycle load.

Viewing the above-mentioned situations, it can be observed that two-first cases may not dangerous; however, shakedown behavior (Figure 2.5(b)) thoroughly exploits the capacity of materials.

Consider a body made of elastic-perfectly plastic materials and is subjected to a load $\mathbf{F}$. The displacement and deformation are assumed to be small enough to
ignore the geometrical change in equilibrium equations. The external load $\mathbf{F}$ are decomposed into two parts including body force $\mathbf{f}$ and surface load $\mathbf{t}$. Denoting $\mathbf{F}_0(\mathbf{f}_0, \mathbf{t}_0)$ for the initial load applying to the structure, thus

$$\mathbf{F} = \lambda \mathbf{F}_0$$

(2.18)

where $\lambda$ is the load multiplier; the load $\mathbf{F}$ is assumed that proportionally increase with $\lambda$. Denoting $\Omega_u$ and $\Omega_t$ for the kinematic and static of body, stress and strain components must satisfy the equilibrium condition, the compatibility condition as well as the kinematic and static boundary conditions completely

$$\sigma + \mathbf{f} = 0 \quad \text{in } V$$  \hspace{1cm} (2.19a)

$$\dot{\epsilon} = \nabla \dot{\mathbf{u}} \quad \text{in } V$$  \hspace{1cm} (2.19b)

$$\mathbf{n}\sigma = \mathbf{t} \quad \text{on } \Omega_t$$  \hspace{1cm} (2.19c)

$$\dot{\mathbf{u}} = \overline{\dot{\mathbf{u}}} \quad \text{on } \Omega_u$$  \hspace{1cm} (2.19d)

where $\dot{\mathbf{u}}$ is the deformation velocity; $\overline{\dot{\mathbf{u}}}$ is the known displacement velocity value; $\mathbf{n}$ is the outward normal of static boundary $\Omega$; $\sigma$ and $\epsilon$ are stress and strain components, respectively.

The development of shakedown analysis bases on the fundamentals of the kinematic and static theorems well-known as the upper bound and lower bound of direct method.

### 2.2.1 Upper bound theorem of shakedown analysis

Using plastic strain field, the kinematic theorem of shakedown analysis was formulated by Koiter \[15\]. Following Koiter, the plastic strain rate $\dot{\epsilon}^p$ is not required to satisfy the compatibility condition at each instant during the cycle, but the total plastic deformation $\Delta \dot{\epsilon}^p$ accumulated after each cycle must satisfy the kinematically compatible condition, it means

$$\Delta \dot{\epsilon}^p = \int_0^T \dot{\epsilon}^p dt \quad \text{in } V$$ \hspace{1cm} (2.20a)

$$\Delta \dot{\epsilon}^p = \nabla \Delta \dot{\mathbf{u}} \quad \text{on } \Omega_u$$ \hspace{1cm} (2.20b)

$$\dot{W}_E = \int_0^T dt \left[ \int_V f \dot{\mathbf{u}} dV + \int_{\Omega_t} \mathbf{t} \dot{\mathbf{u}} d\Omega \right] > 0$$ \hspace{1cm} (2.20c)
Theorem 1. Upper bound theorem of shakedown analysis

1. Shakedown may happen if the following inequality is satisfied

\[
\int_0^T dt \left[ \int_V f \dot{u} dV + \int_{\Omega_t} t \dot{u} d\Omega \right] \leq \int_0^T dt \int_V D(\dot{\epsilon}) dV
\]  

(2.21)

2. Shakedown cannot happen when the following inequality holds

\[
\int_0^T dt \left[ \int_V f \dot{u} dV + \int_{\Omega_t} t \dot{u} d\Omega \right] > \int_0^T dt \int_V D(\dot{\epsilon}) dV
\]  

(2.22)

where the plastic dissipation power \( D_p(\dot{\epsilon}) \) is given by

\[
D_p(\dot{\epsilon}) = \sigma \dot{\epsilon}
\]  

(2.23)

Applying the principle of virtual work, the power generated by external load can be recalculated as

\[
\dot{W}_E = \int_0^T dt \int_V \sigma^E(x, t) \dot{\epsilon}^p dV
\]  

(2.24)

with \( \sigma^E(x, t) \) is the elastic fictitious stress at time \( t \) in the loading domain \( \mathcal{D} \).

Normalizing external work, the upper bound of shakedown load multiplier can be obtained by solving the optimization problem

\[
\lambda^+ = \min \int_0^T dt \int_V D_p(\dot{\epsilon}) dV
\]

s.t

\[
\begin{align*}
\Delta \dot{\epsilon} &= \int_0^T \dot{\epsilon} d\Omega \\
\Delta \dot{\epsilon} &= \Delta \nabla \dot{\mathbf{u}} \quad \text{in } V \\
\Delta \dot{\mathbf{u}} &= 0 \quad \text{on } \Omega_u \\
\dot{W}_E &= 1
\end{align*}
\]

(2.25)

2.2.2 The lower bound theorem of shakedown analysis

Shakedown occurs after several first loading cycle when the plastic strains stop \( (\dot{\epsilon}^p = 0) \) at everywhere within the structure, and it is in need to introduce a fictitious residual stress field \( \rho(\mathbf{x}) \) ensuring that the actual stress field \( \sigma(x, t) \) does not violate
the yield criterion everywhere

\[ \psi [\sigma(x,t)] = \psi [\lambda \sigma^E(x,t) + \rho(x)] \leq 0 \]  \hspace{1cm} (2.27)

where

\[ \sigma(x,t) = \lambda \sigma^E(x,t) + \rho(x) \]  \hspace{1cm} (2.28)

Due to the elastic stress \( \sigma^E(x,t) \) equilibrates to the external load, the residual stress field \( \rho(x) \) must be in self-equilibrium state

\[ \nabla \rho(x) = 0 \quad \text{in} \ V \]  \hspace{1cm} (2.29a)
\[ n \rho(x) = 0 \quad \text{on} \ \Omega_t \]  \hspace{1cm} (2.29b)

The necessary and sufficient conditions for shakedown are given by Melan \[14\] as following theorem.

**Theorem 2. Lower bound theorem of shakedown analysis**

1. Shakedown occurs if there exists a permanent residual stress field \( \rho \), statically admissible, such that

\[ \psi [\lambda \sigma^E(x,t) + \rho(x)] < 0 \]  \hspace{1cm} (2.30)

2. Shakedown will not occur if no \( \rho \) exists such that

\[ \psi [\lambda \sigma^E(x,t) + \rho(x)] \leq 0 \]  \hspace{1cm} (2.31)

Based on above theorem, a statically admissible residual stress field needs to be determine to obtain the maximum load domain \( \lambda^- \Omega \), in where the load multiplier \( \lambda^- \) is the lower bound of the actual factor. Now, the shakedown problem can be considered as maximizing a nonlinear optimization problem

\[ \lambda = \max \lambda^- \]  \hspace{1cm} (2.32)

subject to

\[ \psi [\lambda \sigma^E(x,t) + \rho(x)] \leq 0 \quad \forall t \]  \hspace{1cm} (2.33)
2.2.3 Separated and unified methods

For determining the shakedown limit of structures, there are two popular methods: separated and unified methods. The first procedure assumes that the incremental collapse and alternating plasticity may occur at the same time. In this model, the kinematically admissible strain is decomposed into two parts involving alternating incremental collapse and alternating plasticity. Solving the optimization problem, the upper bound of plastic shakedown load multiplier will be obtained.

This thesis focuses on the unified model for which the static formulation introduced by Melan [14] will be employed. The obstacle caused by the time-dependent variables and integrals can be overcome using the convex-cycle theorems relating to the concept of load domain mentioned following.

2.2.4 Load domain

Shakedown analysis considers structures under \( n_L \) independent loading processes \( \mathcal{P}(t) \) forming a convex polyhedral domain \( \mathcal{D}(x,t) \) so-called load domain with \( n_L \) dimensions and \( (m = 2^{n_L}) \) vertices. The loading path \( \mathcal{D}(x,t) \) can be expressed as a linear combination of loading processes as

\[
\mathcal{P}(t) = \sum_{k=1}^{n_L} \mu_k(t) \mathcal{P}_k^0
\]

where \( \mathcal{P}_k^0 \) is the initial load; the parameter \( \mu_k(t) \) can be given by

\[
\mu_k(t) \in [\mu_k^-, \mu_k^+] , \quad k = 1, 2, \ldots, n_L \quad (2.35)
\]

Solving the optimization problems (2.26) and (2.33), in order to overcome the difficulty generated by the appearance of time-dependent variables and time-dependent integrals, König and Kleiber [185] introduced the convex-cycle theorems as follows

**Theorem 3.** Shakedown will happen over a given load domain \( \mathcal{D} \) if and only if it happens over the convex envelope of \( \mathcal{D} \).

**Theorem 4.** Shakedown will happen over any load path within a given load domain \( \mathcal{D} \) if it happens over a cyclic load path containing all vertices of \( \mathcal{D} \).

Figure 2.6(a) and 2.6(b) illustrate the uses of load cycle for structures subjected
Chapter 2 Fundamentals

Figure 2.6: Loading cycles in shakedown analysis

to two independent loads. Above convex-cycle theorems has shown that it is sufficient to consider only the vertices of the convex polyhedral loading domain instead of time-dependent analysis. The expressions (2.26) and (2.33) for shakedown analysis can be reformulated as

1. Upper bound shakedown analysis

\[
\lambda^+ = \min \left[ \sum_{k=1}^{m} \int_V D_p(\dot{\varepsilon}) dV \right]
\]

s.t

\[
\begin{align*}
\Delta \dot{\varepsilon} &= \sum_{k=1}^{m} \dot{\varepsilon} \\
\Delta \dot{\varepsilon} &= \Delta \nabla \dot{\mathbf{u}} \quad \text{in } V \\
\Delta \dot{\mathbf{u}} &= 0 \quad \text{on } \Omega_t \\
\hat{W}_E &= \sum_{k=1}^{m} \int_V \sigma^E(x, \hat{\mathcal{P}}_k(x)) \dot{\varepsilon}^p dV
\end{align*}
\]

2. Lower bound shakedown analysis

\[
\lambda = \max \lambda^-
\]

s.t

\[
\begin{align*}
\nabla \rho(x) &= 0 \quad \text{in } V \\
n \rho(x) &= 0 \quad \text{on } \Omega_t \\
\psi \left[ \lambda \sigma^E(x, \hat{\mathcal{P}}_k(x)) + \rho(x) \right] &\leq 0 \quad \forall k = 1, 2, \ldots, m
\end{align*}
\]

It is important to note that when there is only one loading point, i.e., \( m = 1 \),
shakedown formulations will be reduced to a limit analysis problem presented in the following section.

2.3 Limit analysis

As above mentioned, limit analysis is a special case of shakedown ones when the structures are subjected to instantaneous loads increasing gradually until the collapse appears. Similar to shakedown analysis, the limit load multiplier $\lambda$ can be determined using one of two opposite formulations based on the bounding theorems.

2.3.1 Upper bound formulation of limit analysis

A kinematically admissible displacement velocity field is assumed. The upper-bound limit analysis of structures can be determined by solving the optimization problem

\[
\begin{align*}
\lambda^+ &= \min_V D_p(\dot{\epsilon})dV \\
\text{s.t} & \begin{cases}
\dot{\epsilon} = \nabla \dot{u} & \text{in } V \\
\dot{u} = 0 & \text{on } \Omega \\
W_E = 1
\end{cases}
\end{align*}
\] (2.40)

For convenience and simplicity, from now, displacement/strain rate or velocity is termed displacement/strain, and the plasticity dissipation as well as the external work relating to displacement velocity are also performed in terms of displacement or strain. The upper bound formulation of limit analysis (2.41) can be rewritten as

\[
\lambda^+ = \min_V D(\epsilon)dV \\
\text{s.t} \begin{cases}
\epsilon = \nabla u & \text{in } V \\
u = 0 & \text{on } \Omega_u \\
W_E = 1
\end{cases}
\] (2.43)
2.3.2 Lower bound formulation of limit analysis

A statically admissible stress field $\sigma$ is assumed. The lower bound limit load multiplier will be obtained if the yield criterion is not violated everywhere, and the static formulation of limit analysis can be expressed as

$$\lambda^- = \max \lambda$$

\[\text{s.t} \begin{cases} \nabla \sigma = 0 & \text{in } V \\ n \sigma = t & \text{on } \Omega_t \\ \psi(\sigma) \leq 0 & \text{in } V \end{cases}\]  \hspace{1cm} (2.45)

In order to obtain the limit and shakedown load multipliers, the formulations (2.26, 2.33) and (2.43, 2.45) can be solved using various optimization tools. In this thesis, the so-called primal-dual interior point algorithm will be utilized. The problems will be cast as second order cone programming (SOCP) and solved using the commercial software package named Mosek integrated in Matlab programming.

2.4 Conic optimization programming

Conic optimization is a sub-field of convex optimization, where linear function is minimized over the intersection of an affine subspace and a convex cone. A set $\mathcal{K}$ in vector space $\mathcal{V}$ is called a cone if

$$\forall x \in \mathcal{K}, \lambda > 0 \Rightarrow \lambda x \in \mathcal{K}$$

where the cone is considered with the following properties

- The cone $\mathcal{K}$ is convex if and only if: $\forall x, x' \in \mathcal{K}, \lambda > 0, \lambda' > 0 \Rightarrow \lambda x + \lambda' x' \in \mathcal{K}$

- The cone $\mathcal{K}$ is non-empty and closed if: $x, x' \in \mathcal{K} \Rightarrow x + x' \in \mathcal{K}$

- The cone $\mathcal{K}$ is pointed if it contains the original point: $x, -x \in \mathcal{K} \Rightarrow x = 0$

Recently, several relevant models of conic programming developed for treatment of convex functions [51] can be listed following
• The non-negative orthant
\[ \mathcal{K} \equiv \mathbb{R}_+^n = \{ x \in \mathbb{R}^n | x_i \geq 0, i = 1, \ldots, n \} \] (2.47)

• The standard second-order cone (Lorentz or ice-cream)
\[ \mathcal{K} \equiv \mathcal{L}_q^n = \{ x \in \mathbb{R}^n | x_1 \geq \|x_{2\to n}\|_{L^2} \} \] (2.48)

• The rotated quadratic cone
\[ \mathcal{K} \equiv \mathcal{L}_r^n = \{ x \in \mathbb{R}^n | x_1 x_2 \geq \|x_{3\to n}\|_{L^2}^2, x_1, x_2 \geq 0 \} \] (2.49)

• The semi-definite cone
\[ \mathcal{K} \equiv \mathcal{S}_+^n = \{ X \in \mathbb{R}^{n \times n} | X \succeq 0, X = X^T \} \] (2.50)

with \( \succeq \) is used to the positive semi-definite matrix.

The dual form \( \mathcal{K}^* \) of the cone \( \mathcal{K} \) can be defined by
\[ x^T y \geq 0, \forall x \in \mathcal{K} \Leftrightarrow y \in \mathcal{K}^* \] (2.51)

and the cone will be self-dual if \( \mathcal{K} = \mathcal{K}^* \).

Following Ciria et al. [175], BenTal and Nemirovski [51], almost yield criterion can be formulated in terms of conic programming consisting of the linear objective and conic constrains. The primal and dual formulations of conic programming can be recalled as

• Primal formulation
\[ \lambda = \min c^T x \]
\[ \text{s.t. } \begin{cases} Ax = b \\ x \in \mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \ldots \times \mathcal{K}_N \end{cases} \] (2.52)
• Dual formulation
\[
\lambda = \max b^T y \\
\text{s.t.} \begin{cases} 
A^T y + s = c \\
x \in K^* = K_1^* \times K_2^* \times \ldots \times K_N^*
\end{cases}
\]
(2.53)
where \(x, y \in \mathbb{R}^n\) are the optimization variable vectors; \(A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n\) are the parameters; \(s \in \mathbb{R}^n\) denotes the additional variables for problems.

Several common conic programming widely used for solving the optimization problems can be be now expressed as

• Linear programming (LP): \(K \equiv \mathbb{R}_+^n\)
• Second-order cone programming (SOCP): \(K \equiv \mathbb{L}_q^n\) or \(K \equiv \mathbb{L}_r^n\)
• Semi-definite programming (SDP): \(K \equiv \mathbb{S}_+^n\)

where LP is the particular case of SOCP, and both models can be cast as a SPD.

2.5 Homogenization theory

In homogenization analysis, material is considered under the connection between two scales involving down-scale from macro-level to micro-level (localization) and up-scale from micro-level to macro-level (globalization). For macro-to-micro translation, the macroscopic features are transformed to micro-structure as the boundary conditions. By the opposite direction, the microscopic properties are transformed to macro-structure via the average.

Let consider a heterogeneous micro-based cell \(\Omega \in \mathbb{R}^2\) so-called the representative volume element (RVE) at every material point \(x \in V\), where \(V \in \mathbb{R}^2\) denotes the heterogeneous macroscopic-continuum. The micro-structure is subjected to the body force \(f\), the surface load \(t\) on the static boundary \(\Gamma_t\) and fixed by the displacement field \(u\) on the kinematic boundary \(\Gamma_u\). The material response of macro-structure is determined by solving the macro-micro transitions problems, where the RVE size plays an important role in the analysis. The RVE dimension must be significantly great to describe the material properties, but significantly small to ensure that the heterogeneity is separately identified. Actually, the size of microscopic base
cell is very small compared with those of macro-scale; therefore, the body force \( f \) can be neglected in the micro-scale problem.

![Homogenization technique: correlation between macro- and micro-scales](image)

Figure 2.7: Homogenization technique: correlation between macro- and micro-scales

The micro-scale problem can be treated as the boundary value one in solid mechanics, where the overall strain \( E \) are transferred to micro-structure in form of kinematic boundary constrains. At microscopic scale, the local fields is decomposed into two parts: mean term and fluctuation term. Denoting \( X \) for the positional matrix of each material point in the computational domain, the local displacement, strain and stress are now given by

\[
\begin{align*}
\mathbf{u}(x) &= \mathbf{E} \cdot X + \tilde{\mathbf{u}}(x) \quad (2.54a) \\
\epsilon(x) &= \mathbf{E} + \tilde{\epsilon}(x) \quad (2.54b) \\
\sigma(x) &= \Sigma + \tilde{\sigma}(x) \quad (2.54c)
\end{align*}
\]

where \( \Sigma \) is the overall stress; \( \tilde{\mathbf{u}}(x) \), \( \tilde{\epsilon}(x) \) and \( \tilde{\sigma}(x) \) denote the fluctuation parts of displacement, strain and stress rate.

For the purpose of enforcing the boundary condition, this study uses the the most efficient in terms of convergence rate so-called periodic procedure, where there are the periodicity of fluctuation displacement field and anti-periodicity of traction field on RVE boundary

\[
\begin{align*}
\tilde{\mathbf{u}}(x^+) &= \tilde{\mathbf{u}}(x^-), \quad \text{on } \Gamma_u \quad (2.55a) \\
\mathbf{t}(x^+) &= -\mathbf{t}(x^-), \quad \text{on } \Gamma_t \quad (2.55b)
\end{align*}
\]

where \( \tilde{\mathbf{u}}(x^+) \) and \( \tilde{\mathbf{u}}(x^-) \) are the fluctuation displacement field, \( \mathbf{t}(x^+) \) and \( \mathbf{t}(x^-) \) are
the traction field of positive and negative boundaries, respectively.

Note that, regarding to the periodic characteristic of the fluctuation terms, the average of \( \tilde{\epsilon}(x) \) and \( \tilde{\sigma}(x) \) over the RVE should vanish, it means

\[
\langle \tilde{\epsilon} \rangle = 0; \quad \langle \tilde{\sigma} \rangle = 0
\]  

where the operation \( \langle . \rangle \) stands the volume average of fields over the RVE. Therefore, the macroscopic quantities can be calculated from the microscopic ones by the average relations

\[
\begin{align*}
E &\equiv \langle \epsilon \rangle = \frac{1}{|\Omega|} \int_{\Omega} \epsilon \mathrm{d}\Omega \\
\Sigma &\equiv \langle \sigma \rangle = \frac{1}{|\Omega|} \int_{\Omega} \sigma \mathrm{d}\Omega
\end{align*}
\]

herein, \( |\Omega| \) denotes the area of RVE.

In direct analysis, for any admissible velocity and stress field satisfying the periodic and anti-periodic conditions on boundary, the principle of macroscopic virtual work can be expressed as

\[
\langle \sigma : \epsilon \rangle = \Sigma : E
\]

### 2.6 The iRBF-based mesh-free method

As mentioned in the beginning of the chapter, iRBF method is the key numerical scheme for solving all problems in the thesis. The smooth function \( u(x) \) can be approximated based on a given set of \( N \) scattered nodes and the iRBF method as

\[
u^h(x) = \sum_{I=1}^{N} \Phi_I(x)u_I
\]

where \( \Phi_I(x) \) is the iRBF shape function; \( u_I \) denotes the nodal values. In here, the iRBF can be understand as the integrated or indirect radial basis functions. The reason for which iRBF method is called indirect procedure is the strategy to construct the shape function clarified following.
2.6.1 iRBF shape function

In this thesis, the RBF functions will be employed to construct the second-order derivative of shape function, then the first-order and the original functions will be calculated using the integrals as

\[
    u^h_{i\alpha\beta}(\mathbf{x}) = \sum_{I=1}^{N} g_I(\mathbf{x}) a_I = \mathbf{R}_2(\mathbf{x}) \mathbf{a}
\]

\[
    u^h_\alpha(\mathbf{x}) = \int \sum_{I=1}^{N} g_I(\mathbf{x}) a_I \, d\beta + C_1 = \sum_{I=1}^{N+n_1} R_{1I}(\mathbf{x}) a_I = \mathbf{R}_1(\mathbf{x}) \mathbf{a}
\]

\[
    u^h(\mathbf{x}) = \int\int \sum_{I=1}^{N} g_I(\mathbf{x}) a_I \, d\beta \, dx_\alpha + C_1 x_j + C_2 = \sum_{I=1}^{N+n_2} R_{0I}(\mathbf{x}) a_I = \mathbf{R}_0(\mathbf{x}) \mathbf{a}
\]

where \( C_1 \) and \( C_2 \) are the integral constants; \( n_1 \) and \( n_2 \) represent number of integral constants (\( n_2 = 2n_1 \)); \( \mathbf{a} \) is the vector consisting the unknowns.

\[
    \mathbf{R}_2(\mathbf{x}) = [g_1(\mathbf{x}), g_2(\mathbf{x}), ..., g_N(\mathbf{x}), 0, ..., 0]_{n_2}
\]

\[
    \mathbf{R}_1(\mathbf{x}) = [R_{11}(\mathbf{x}), R_{12}(\mathbf{x}), ..., R_{1(N+n_1)}(\mathbf{x}), 0, ..., 0]_{n_1}
\]

\[
    \mathbf{R}_0(\mathbf{x}) = [R_{01}(\mathbf{x}), R_{02}(\mathbf{x}), ..., R_{0(N+n_2)}(\mathbf{x})]
\]

with \( R_{0i}, \ R_{1i} \) can be found in [152, 153].

This thesis uses the multiquadric (MQ) basis function well-known as the best iRBF function in terms of accuracy

\[
    g_I(\mathbf{x}) = \sqrt{r_I^2(\mathbf{x}) + (\alpha_s d_I)^2}
\]

where \( r_I(\mathbf{x}) \) is the radius of node \( I \) and other ones in its influent domain; \( d_I \) is the minimum distance measured form node \( I \) to its neighbours; \( \alpha_s > 0 \) is the dimensionless factor used to control the shape parameter \( \alpha_s d_I \).

Estimating the function at the set of \( N \) scattered points, Equation (2.62) can be
rewritten in terms of matrix form as

\[ u = R_0 a \]  

(2.65)

where

\[
R_0 = \begin{bmatrix}
\cdots & \cdots & \cdots & \cdots \\
R_{01}(x_k) & R_{02}(x_k) & \cdots & R_{0(N+n_2)}(x_k) \\
\cdots & \cdots & \cdots & \cdots 
\end{bmatrix}
\]  

(2.66)

As a result, vector \( a \) can be expressed via the nodal values \( u = [u_1 \ u_2 \ \ldots \ u_N] \) as

\[ a = R_0^{-1} u = \hat{\Psi}_{Ik} u \]  

(2.67)

Substituting \( a \) into Equations (2.60) - (2.62), the approximate function and its derivatives are recalculated as

\[ u^h(x) = R_0(x) \hat{\Psi}_{Ik} u = \Phi u \]  

(2.68a)

\[ u^h_{,\alpha}(x) = R_1(x) \hat{\Psi}_{Ik} u = \Phi_{,\alpha} u \]  

(2.68b)

\[ u^h_{,\alpha\beta}(x) = R_2(x) \hat{\Psi}_{Ik} u = \Phi_{,\alpha\beta} u \]  

(2.68c)

where the shape function and its derivatives can be defined by

\[
\Phi_s(x) = \sum_{I=1}^{N} R_{0l}(x) \hat{\psi}_{Is}
\]  

(2.69a)

\[
\Phi_{s,\alpha}(x) = \sum_{I=1}^{N} R_{1l}(x) \hat{\psi}_{Is}
\]  

(2.69b)

\[
\Phi_{s,\alpha\beta}(x) = \sum_{I=1}^{N} g_I(x) \hat{\psi}_{Is}
\]  

(2.69c)

with \( \hat{\psi}_{Is} \) is the element \((I, s)\) within matrix \( R_0^{-1} \).

The iRBF method overcomes an important obstacle of almost meshless methods in enforcing the essential boundary condition caused by the lack of Kronecker-delta property. Moreover, the high-order shape function obtained owing to the integration also helps to improve the computational aspect of numerical approach.
2.6.2 The integrating constants in iRBF approximation

In indirect RBF formulation, the approximate function \( u^h(x) \) is calculated by the multiple integration; therefore, after each step, the new constants \( C_1 \) and \( C_2 \) appear. In this thesis, the integrating constants will be computed using the similar way to those carried out for the approximate function \( u^h(x) \), it means the multiple integrals will be utilized as

\[
C^h_{j,k}(x) = \sum_{I=1}^{N} g_I(x) \\
C^h_{j}(x) = \int \sum_{I=1}^{N} g_I(x) dx_k + \hat{C}_1 \\
C^h(x) = \iint \sum_{I=1}^{N} g_I(x) dx_k dx_j + \hat{C}_1 x_j + \hat{C}_2
\]  

Herein, two new constant \( \hat{C}_1 \) and \( \hat{C}_2 \) will occur, and strictly, those must be calculated. However, thank to the constants \( C_1 \) and \( C_2 \) have been approximated,
for simplify, $\hat{C}_1$ and $\hat{C}_2$ can be ignored without effect on the approximation of the iRBF shape function.

It should be note that, in RPIM approximation, the radial basis cannot produce the linear polynomials exactly; consequently, the polynomials must be added into the basis functions to ensure the consistency of shape function. In other words, that makes sure the reproduction of the approximated field, and hence, helps to pass the standard patch tests, more details can be found in [156]. In this study, with the use of integrals when constructing shape functions, the constants $C_1$ and $C_2$ are generated evidently, making the iRBF approximation passes the patch tests naturally. However, it’s worth noting that the patch test is neither sufficient nor necessary for convergence of numerical solutions [186], and many finite elements are widely used in FEM packages without passing this test [156].

2.6.3 The influence domain and integration technique

In order to evaluate the efficiency of a numerical method, it is necessary to consider not only the accuracy, stability and convergence of solutions but also the computational cost. In mesh-free method, that depends on the influence or support domain and the technique to handle the integration.

The influence or support domain is defined as an area where a node or a point exerts an influence upon, but it is necessary to distinguish clearly between support and influence domains in meshless methods. The concept of support domain is used for the purpose of interpolating a value at a point. That domain is usually a small local field including a number of nodes in the problem domain. As an alternative way to select nodes for interpolation, the influence domain is defined for each node in the problem domain. In mesh-free methods, the influence domain can be global or local, and the density of the nodes depends on the accuracy requirement of the analysis and the resources available. The global domain enclosing all nodes in the problem domain ensures the continuity and ability to approximate and interpolate functions; hence the highly accuracy solutions can be obtained. However, using lots of nodes, the matrix of shape function and constrains in the problem will become densely increasing the expense of computation. In the opposite direction, the compact domain makes sure the local property and reduces the computational cost significantly, but it requires enough nodes to avoid the inaccuracy and singularity when approximating the shape function and interpolating nodal values. There are
several technique to determine the influence domain in mesh-free methods. The shapes of the domain mostly used are circular or rectangular. Since this thesis uses the radial basis function, the circular domain will be utilized as shown in Figure 2.9(a). The size of this domain can be operated using formulae (1.1) presented in previous chapter.

![Figure 2.9: The influence domain and representative domain of nodes](image)

In several mesh-free methods, the background cells can be utilized to create the Gauss points where the computation will be implemented on. However, as discussed in previous chapters, this work does not ensure the truly meshless feature. In addition, a very fine mesh generating a number of Gauss points is required to obtained the good solutions, and hence the cost increases. Several meshless procedures in strong form use the collocation method, the constraints will be enforced and satisfied directly at scattered nodes. Consequently, the expense for the computation can be reduced significantly. Beside the advantage owing to the simplicity in implementation, the well-known drawback of strong form methods is the lack of stability and accuracy. In this thesis, a weak form using the stability conforming nodal integration (SCNI) technique [166] is employed, all constrains will be directly imposed at nodes instead of Gauss points. This scheme not only keeps the size of problems in a minimum but also ensures to obtain the solutions with high accuracy, stability and convergence.

Using SCNI technique, each node will has a integration area so-called representative domain $\Omega_J$. For convenience, this domain can be determined using Voronoi
diagram as illustrated in Figure 2.9(b). The smoothed version of strain rate at node \( \tilde{\epsilon}_{ij}(x_J) \) can be calculated by

\[
\tilde{\epsilon}_{ij}(x_J) = \int_{\Omega_J} \epsilon_{ij}(x) \varphi(x, x - x_J) d\Omega
\]  
(2.71)

where \( \varphi \) is the smoothed function satisfying the condition

\[
\varphi > 0 \quad \text{and} \quad \int_{\Omega_J} \varphi d\Omega = 1
\]  
(2.72)

For simplicity, assuming that the function \( \varphi \) is a constant over the representative domain

\[
\varphi(x, x - x_J) = \begin{cases} 
\frac{1}{A_J} & \text{if } x \in \Omega_J, \\
0 & \text{if } x \notin \Omega_J.
\end{cases}
\]  
(2.73)

where \( A_J \) is the area of domain \( \Omega_J \).

Substituting \( \varphi \) in (2.73) into (2.71), then applying the Green’s theorem to transfer the integral over domain to the line integral, the smoothed strain rate can be rewritten as

\[
\tilde{\epsilon}_{ij}(x_J) = \frac{1}{A_J} \int_{\Omega_J} \frac{1}{2} (u_{ij}^h + u_{ji}^h) d\Omega = \frac{1}{2A_J} \oint_{\Gamma_J} (u_i^h n_j + u_j^h n_i) d\Gamma
\]  
(2.74)

where \( \Gamma_J \) is the boundary of the representative domain; \( u_i \) and \( u_j \) are the displacement components; \( n \) is the outward normal of edges bounding domain \( \Omega_J \) as Figure 2.10.

In the numerical implementation, the smoothed strain \( \tilde{\epsilon}(x_J) \) can be expressed via the relation to the displacements according to the compatible condition as follow

\[
\tilde{\epsilon}(x_J) = \tilde{B}d
\]  
(2.75)

where \( d \) is the displacement vector; \( \tilde{B} \) is the displacement-strain matrix including
the smoothed derivatives of shape function

\[
\hat{\Phi}_{I,\alpha}(x_j) = \frac{1}{A_J} \int_{\Gamma_J} \Phi_I(x_J)n_\alpha(x)d\Omega \\
= \frac{1}{2A_J} \sum_{k=1}^{ns} \left(n^{k}_{\alpha}L^{k} + n^{k+1}_{\alpha}L^{k+1}\right) \Phi_I(x^{k+1}_J)
\]  

(2.76)

where \( ns \) is number of edges; \( \Phi_I(x^k_J) \) and \( \Phi_I(x^{k+1}_J) \) are the shape function relating to two end of the edge \( \Gamma^k \) of Voronoi; \( L^k \) and \( n^k \) are the length and the normal of edge \( \Gamma^k \).

In several problems, the high-order derivatives of shape function may be in need due to the variables can be the high-order derivatives of displacement, for example in bending plate problem, the curvature variables \( \kappa(x) \) are the second-order derivative of the deflection \( w(x) \). Similarly, the smoothed version of second-order derivatives of shape function can be calculated from the first-order ones as

\[
\hat{\Phi}_{I,\alpha\beta}(x_J) = \frac{1}{4A_J} \int_{\Gamma_J} \Phi_I(x_J)(n_\beta(x) + \Phi_{I,\beta}(x_J)n_\alpha(x))d\Omega \\
= \frac{1}{4A_J} \sum_{k=1}^{ns} \left(n^{k}_{\beta}L^{k} + n^{k+1}_{\beta}L^{k+1}\right) \Phi_I(x^{k+1}_J) \\
+ \frac{1}{4A_J} \sum_{k=1}^{ns} \left(n^{k}_{\alpha}L^{k} + n^{k+1}_{\alpha}L^{k+1}\right) \Phi_{I,\beta}(x^{k+1}_J)
\]  

(2.77)

with \( \Phi_{I,\alpha}(x) \) and \( \Phi_{I,\beta}(x) \) are the first-order derivatives of shape function \( \Phi_I(x) \) relating to variables \( \alpha \) and \( \beta \).
Chapter 3

Displacement and equilibrium mesh-free formulation based on integrated radial basis functions for dual yield design

3.1 Introduction

This chapter presents an application of iRBF-based mesh-free method for plane structures at limit state using both of kinematic and static formulations. A lower bound on the actual limit load of a structure or body can be achieved by using the static theorem and approximated stress fields, while the upper bound is obtained as a result of combining displacement-based model and kinematic theorem \[93\]. In the static yield design formulation, the assumed stress fields are often expressed in terms of nodal stress values. In the framework of equilibrium finite elements, these approximated fields are also required to satisfy a priori equilibrium conditions within elements and at their interfaces \[36, 58, 93, 187, 188\]. Due to these additional conditions, construction of such fields is often difficult. Compared with the equilibrium models, the displacement formulation is more popular. This may be because of the facts that the internal compatibility condition can be satisfied straightaway in the assembly scheme, and that essential (kinematic) boundary conditions can be enforced directly.

Unlike FEM, mesh-free methods does not encounter the obstacle of enforcing the equilibrium conditions thank to their independence to elements, and hence the displacement as well as equilibrium formulations can be used easily. The EFG method, one of the most widely used mesh-free methods, has been applied successfully to

the framework of yield design problems [90, 92–95], showing that the method is, in
general, well suited for yield design problems and that accurate solutions can be ob-
tained with a minimal computational cost. However, a typical limitation of the EFG
method is that its shape functions do not hold Kronecker delta property, leading to
difficulty in enforcing essential boundary conditions. An other mesh-free procedure
so-called NEM has been applied for limit analysis in [88, 89]. Possessing a weak
form of Kronecker-delta property at the boundary, NEM shows more advantage
than EFG in imposing the boundary conditions. Using the stricter Kronecker-delta
property compared with NEM, RPIM has been also extent to this area in [96].

The aim of this study is to investigate the performance of the integrated radial
basis function-based mesh-free method in the framework of yield design problems.
The iRBF approach will be employed to approximate both displacement and stress
fields. Multiquadric iRBF method generally results in a high order approximation of
the displacement fields, and hence volumetric locking phenomena in the kinematic
yield design formulation can be prevented. Moreover, the stress fields constructed
based on iRBF are smooth over the entire problem domain, and consequently there
is no need to enforce continuity conditions at interfaces within the problem domain.
With the use of iRBF-approximated stress fields the strong-form of equilibrium
equations can be satisfied in a point-wise manner using a collocation method. In
addition, the iRBF-based approximation possesses the Kronecker delta property as
in RPIM, but the order of iRBF shape function is higher than RPIM ones when
using similar basis function. As a result, kinematic and static boundary conditions
can be imposed as easily as in the finite element method. Finally, the kinematic
and static formulations based on iRBF discretization are formulated as a conic
optimization problem, ensuring that they can be solved using available efficient
solvers.

3.2 Kinematic and static iRBF discretizations

Consider an elastic-plastic body of area $\Omega \in \mathbb{R}^2$, with fixed boundary $\Gamma_u$ and
free portion $\Gamma_t$ such that $\Gamma_u \cup \Gamma_t = \Gamma$, $\Gamma_u \cap \Gamma_t = \emptyset$, and is subjected to a body force
$f$ in $\Omega$ and surface traction $t$ on $\Gamma_t$. The structure is investigated in both kinemat-
ically and statically admissible spaces. The iRBF-based mesh-free method will be
utilized to approximate the statically admissible stress field as well as the kinemat-
ically displacement velocity field. As presented in previous chapter, for simplify, the
displacement velocity field will be briefly called displacement field.

3.2.1 iRBF discretization for kinematic formulation

For the upper bound analysis at limit state, the displacement field can be approximated via the nodal values as

$$u^h(x) = \sum_{I=1}^{N} \Phi_I(x) \begin{bmatrix} u_I \\ v_I \end{bmatrix}$$

(3.1)

where, $\Phi_I(x)$ denotes the iRBF shape function.

The strain rate are calculated by

$$\epsilon = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \sum_{I=1}^{N} \Phi_{I,x}(x) \\ 0 \\ \sum_{I=1}^{N} \Phi_{I,y}(x) \end{bmatrix} \begin{bmatrix} u_I \\ v_I \end{bmatrix} = B(x)d$$

(3.2)

where $d$ is the nodal displacement vector; $B$ is called displacement-strain matrix and given by

$$d^T = [u_1, u_2, ..., u_n, v_1, v_2, ..., v_n]$$

(3.3a)

$$B = \begin{bmatrix} B_{xx} \\ B_{yy} \\ B_{xy} \end{bmatrix} = \begin{bmatrix} \Phi_{1,x} & \Phi_{2,x} & ... & \Phi_{N,x} \\ 0 & 0 & ... & 0 \\ \Phi_{1,y} & \Phi_{2,y} & ... & \Phi_{N,y} \end{bmatrix}$$

(3.3b)

For von Mises yield criterion, the dissipation power can be formulated as

$$D_p(\epsilon) = \int_{\Omega} \sigma_p \sqrt{\epsilon^T \Theta \epsilon} = \sum_{I=1}^{N} \sigma_p A_I \sqrt{(B_I d)^T \Theta B_I d}$$

(3.4)

where $\sigma_p$ is the yield stress of materials; $A_I$ is area of the representative domain
Chapter 3. Displacement and equilibrium mesh-free formulation based on iRBF for dual yield design

$I^{th}$, e.g. Voronoi cells; $N$ denotes number of nodes in problem domain and

$$
\Theta = \frac{1}{3} \begin{bmatrix}
4 & 2 & 0 \\
2 & 4 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

for plane stress problem \hfill (3.5)

or

$$
\Theta = \begin{bmatrix}
1 & -1 & 0 \\
-1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

for plane strain problem. \hfill (3.6)

Using SOCP, a sum of norm can be employed to calculate the internal dissipation work as

$$
D_p = \sum_{I=1}^{N} \sigma_p A_I \| \rho_I \| \hfill (3.7)
$$

with $\rho_I$ denotes the additional variables defined by

$$
\rho_I = \begin{cases}
\begin{bmatrix}
\rho_1 \\
\rho_2 \\
\rho_3
\end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix}
2 & 0 & 0 \\
1 & \sqrt{3} & 0 \\
0 & 0 & 1
\end{bmatrix} B_{Id} & \text{for plane stress;} \\
\begin{bmatrix}
\rho_1 \\
\rho_2
\end{bmatrix} = \begin{bmatrix}
B_{xxI}d - B_{yyI}d \\
2B_{xyI}d
\end{bmatrix} & \text{for plane strain.}
\end{cases}
\hfill (3.8)
$$

Now, the optimization can be rewritten as follow

$$
\lambda^+ = \min \sum_{I=1}^{N} \sigma_p A_I \| \rho_I \| \hfill (3.9)
$$

s.t

$$
\begin{cases}
d = 0 \text{ on } \Gamma_u \\
F(d) = 1
\end{cases}
\hfill (3.10)
$$

Introducing additional variables $t_1, t_2, ..., t_N$, problem (3.10) can be reformulated
in form of second-order cone programming as

$$\lambda^+ = \min_{I=1}^{N} \sigma_p A_I t_I$$  \hspace{1cm} (3.11)

s.t

$$\begin{aligned}
\mathbf{d} &= 0 \text{ on } \Gamma_u \\
F(\mathbf{d}) &= 1 \\
\|\rho_I\| &\leq t_I, \ I = 1, 2, ..., N
\end{aligned}$$ \hspace{1cm} (3.12)

Note that for plane strain problems, incompressibility conditions, $\Delta^T \epsilon = 0$ with $\Delta^T = [1, 1, 0]$, must be introduced. If low-order displacement approximations are used, volumetric locking phenomena in the kinematic formulations associated with the von Mises may occur due to these incompressibility conditions. However, here the iRBF method results in high-order displacement fields, and hence volumetric locking problem can be prevented. Moreover, it is evident that the size of optimization problem (3.12) depends on the number of integration points to be used. In this study, the nodal integration technique is used, and hence the size of the resulting optimization problem is kept to be minimum.

### 3.2.2 iRBF discretization for static formulation

While in the upper bound formulation the displacement fields are approximated, here the stress fields need to be approximated. With the use of the iRBF method, approximations of these stress fields can be presented as

$$\mathbf{\sigma}^h(\mathbf{x}) = \begin{bmatrix} \sigma^h_{xx} \\ \sigma^h_{yy} \\ \sigma^h_{xy} \end{bmatrix} = \sum_{I=1}^{N} \Phi_I(\mathbf{x}) \begin{bmatrix} \sigma_{xxI} \\ \sigma_{yyI} \\ \sigma_{xyI} \end{bmatrix} = \mathbf{C} \mathbf{s}$$ \hspace{1cm} (3.13)

where

$$\mathbf{s}^T = \begin{bmatrix} \sigma_{xx1}, \ldots, \sigma_{xxN}, \sigma_{yy1}, \ldots, \sigma_{yyN}, \sigma_{xy1}, \ldots, \sigma_{xyN} \end{bmatrix}$$ \hspace{1cm} (3.14a)

$$\mathbf{C} = \begin{bmatrix} \Phi_1 & \cdots & \Phi_N & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \Phi_1 & \cdots & \Phi_N & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \Phi_1 & \cdots & \Phi_N \end{bmatrix}$$ \hspace{1cm} (3.14b)
These approximated stress fields must be ensured to be statically admissible, meaning that equilibrium and continuity conditions within elements and on their boundary must be satisfied. While the strong form of equilibrium equations can be treated using collocation method, its equivalent weak form (involving integrals) is often handled using the weighted residual method. The strong-form method is simple and fast, and hence the collocation method using the iRBF will be considered in this study. The equilibrium equations can be imposed at \( N \) nodes, and are expressed as

\[
\begin{align*}
A_1 \sigma_1 + A_2 \sigma_3 &= 0 \\
A_1 \sigma_3 + A_2 \sigma_2 &= 0
\end{align*}
\tag{3.15}
\]

with

\[
\begin{align*}
A_1 &= \begin{bmatrix}
\cdots & \cdots & \cdots & \cdots \\
\Phi_{1,x}(x_k) & \Phi_{2,x}(x_k) & \cdots & \Phi_{N,x}(x_k) \\
\cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}_{N \times N} \\
A_2 &= \begin{bmatrix}
\cdots & \cdots & \cdots & \cdots \\
\Phi_{1,y}(x_k) & \Phi_{2,y}(x_k) & \cdots & \Phi_{N,y}(x_k) \\
\cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}_{N \times N}
\end{align*}
\tag{3.16a}
\tag{3.16b}
\]

and

\[
\begin{align*}
\sigma_1 &= [\sigma_{x1}, \sigma_{x2}, \cdots, \sigma_{xN}]^T \\
\sigma_2 &= [\sigma_{y1}, \sigma_{y2}, \cdots, \sigma_{yN}]^T \\
\sigma_3 &= [\sigma_{xy1}, \sigma_{xy2}, \cdots, \sigma_{xyN}]^T
\end{align*}
\tag{3.17a}
\tag{3.17b}
\tag{3.17c}
\]

Additionally, the approximated stress fields must belong to a convex domain, \( \mathcal{B} \). In other words, these stress fields must satisfy the following second-order cone constraints obtaining from the von Mises criterion

\[
\sigma^h(x) \in \mathcal{B}
\tag{3.18}
\]
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with

\[ \mathcal{B} \equiv \{ \rho \in \mathbb{R}^3 \mid \rho_1 \geq \| \rho_{2 \rightarrow 4} \|_L = \sqrt{\rho_2^2 + \rho_3^2 + \rho_4^2} \} \quad \text{for plane stress} \quad (3.19a) \]
\[ \mathcal{B} \equiv \{ \rho \in \mathbb{R}^3 \mid \rho_1 \geq \| \rho_{2 \rightarrow 3} \|_L = \sqrt{\rho_2^2 + \rho_3^2} \} \quad \text{for plane strain} \quad (3.19b) \]

where

\[ \rho_1 = \sigma_p \]
\[ \rho_{2 \rightarrow 4} = \begin{bmatrix} \rho_2 \\ \rho_3 \\ \rho_4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ -1 & \sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{3} \end{bmatrix} \mathbf{C}_s \quad \text{for plane stress} \quad (3.20) \]
\[ \rho_{2 \rightarrow 3} = \begin{bmatrix} \rho_2 \\ \rho_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \mathbf{C}_{xx} - \mathbf{C}_{yy} \\ 2\mathbf{C}_{xy} \end{bmatrix} \mathbf{s} \quad \text{for plane strain} \]

Hence the static yield design formulation can now be expressed as

\[ \lambda = \max \; \lambda^- \quad (3.21) \]
\[ \text{s.t} \begin{cases} A_1 \sigma_1 + A_2 \sigma_2 \in \Omega \\ A_1 \sigma_3 + A_2 \sigma_2 \in \Omega \\ \rho^k \in \mathcal{L}^k \quad k = 1, 2, \ldots, N_p \end{cases} \quad (3.22) \]

and accompanied by appropriate boundary conditions.

It should be emphasized that in the present static formulation equilibrium equations and yield criterion are enforced at nodes only, and therefore the strict property of the lower bound \( \lambda^- \) is not guaranteed. However, using a fine nodal distribution one can hope to achieve a reliable approximated lower bound on the actual limit load multiplier. Moreover, by enforcing the equilibrium equations and yield criterion at nodes only the number of constraints in optimization problem (3.22) is kept to be minimum, and hence the presented static method is computationally inexpensive.

The whole numerical implementations of both upper and lower bound approaches are illustrated by flow chart shown in Figure 1.1.
3.3 Numerical examples

The described procedures are tested by their application to solve various problems for which, in most cases, exact and numerical solutions are available. Upper and lower bound solutions based on direct radial basis function (dRBF) are also carried out for comparison purpose. Optimization problems (3.12) and (3.22) are implemented in the Matlab environment. Mosek optimization solver version 6.0 is used to solve the conic optimization problem obtained (using a 2.8 GHz Intel Core i5 PC running Window 7).

3.3.1 Prandtl problem

The first example is the classical punch problem presented in [189], as shown in Figure 3.1. Due to symmetry, a rectangular region of dimensions $B = 5$ and $H = 2$ is considered. Appropriate displacement and stress boundary conditions are enforced as shown in Figure 3.2. For a load of $2\tau_0$, the analytical limit multiplier is $\lambda = 2 + \pi = 5.142$.

![Figure 3.1: Prandtl problem](image)

Approximations of upper and lower bounds on the actual limit load for both dRBF and iRBF methods with various nodal discretizations are reported in Table 3.1. From these results, it can be seen that for both kinematic and static formulations the iRBF-based method can provide more accurate solutions than the dRBF-based method. Convergence analysis and relative errors in collapse multipliers versus number of variables are also shown in Figures 3.3(a) and 3.3(b), respectively. It should be stressed that the mean values of upper and lower approximations obtained using the iRBF-based numerical procedures are in excellent agreement with the analytical solutions for all nodal discretizations, as shown in Figure 3.3, with less than 0.4% even for coarse nodal distribution. Furthermore, as mentioned, the
present procedure cannot theoretically provide strict lower bound solutions, it is evident that all approximated lower bound results are below the exact value.

Figure 3.2: Prandtl problem: approximation displacement and stress boundary conditions

Note that in the kinematic formulation, volumetric (or isochoric) locking often occurs when adding the incompressibility condition to the low-order displacement based yield design problem. The volumetric locking behavior of the Prandtl yield design problem has been studied in [78, 95]. In these papers, it has been demonstrated that when smoothed strains were used, the volumetric locking problem can be eliminated. Here, we have shown that the iRBF method used in combination with direct nodal integration can remove such the volumetric locking behavior and also result in stable and accurate solutions.

In Table 3.2 the solutions obtained using the present methods with 2560 nodes are compared with those obtained previously by different yield design approaches using FEM, smoothed finite element (SFEM) and EFG simulations. In general, the present solutions are close to results in the literature. Considering upper solutions,
Table 3.1: Prandtl problem: upper and lower bound of collapse multiplier

<table>
<thead>
<tr>
<th>Nodes</th>
<th>dRBF</th>
<th>iRBF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Upper bound</td>
<td>Lower-bound</td>
</tr>
<tr>
<td></td>
<td>$\lambda^+$</td>
<td>$\lambda^-$</td>
</tr>
<tr>
<td>40</td>
<td>6.857</td>
<td>2.761</td>
</tr>
<tr>
<td>160</td>
<td>5.548</td>
<td>4.800</td>
</tr>
<tr>
<td>360</td>
<td>5.289</td>
<td>5.042</td>
</tr>
<tr>
<td>640</td>
<td>5.211</td>
<td>5.042</td>
</tr>
<tr>
<td>1000</td>
<td>5.189</td>
<td>5.125</td>
</tr>
</tbody>
</table>

$e$ (%) - relative error

Table 3.2: Prandtl problem: comparison with previous solutions

<table>
<thead>
<tr>
<th>Author</th>
<th>Approach</th>
<th>Collapse load multiplier</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\lambda^+$</td>
</tr>
<tr>
<td>Present method, $i$RBF</td>
<td>Kinematic, static</td>
<td>5.146</td>
</tr>
<tr>
<td>Present method, $d$RBF</td>
<td>Kinematic, static</td>
<td>5.159</td>
</tr>
<tr>
<td>Makrodimopoulos and Martin [190], FEM</td>
<td>Kinematic, static</td>
<td>5.148</td>
</tr>
<tr>
<td>Vicente da Silva and Antao [191], FEM</td>
<td>Kinematic</td>
<td>5.264</td>
</tr>
<tr>
<td>Sloan and Kleeman [192], FEM</td>
<td>Kinematic</td>
<td>5.210</td>
</tr>
<tr>
<td>Le et al. [78], CS-FEM</td>
<td>Kinematic</td>
<td>5.143</td>
</tr>
<tr>
<td>Le et al. [95], EFG</td>
<td>Kinematic</td>
<td>5.147</td>
</tr>
<tr>
<td>Capsoni and Corradi [63], FEM</td>
<td>Mixed formulation</td>
<td>5.240</td>
</tr>
</tbody>
</table>

Figure 3.3: Bounds on the collapse multiplier versus the number of nodes and variables

(a) Bounds on the collapse multiplier

(b) Relative error in collapse multipliers

Figure 3.3: Bounds on the collapse multiplier versus the number of nodes and variables
the result obtained using the iRBF method is slightly lower than the one obtained using the EFG mesh-free method with the same nodal discretization [95].

3.3.2 Square plates with cutouts subjected to tension load

Next, two thin square plates with a central square cutout and a thin crack subjected to a uniform tension load, as shown in Figure 3.4, are considered. These problems have been investigated numerically by finite elements [193, 194], symmetric Galerkin boundary elements [195], and mesh-free methods [88, 90]. Owing to the symmetry, only the top-right quarter of plates is modeled, as shown in Figure 3.5. Uniform nodal distribution is used to discretize the computational domain, see Figure 3.6.

![Figure 3.4: Thin square plates](image)

![Figure 3.5: The upper-right quarter of plates](image)
Chapter 3. Displacement and equilibrium mesh-free formulation based on iRBF for dual yield design

Limit load multipliers obtained using uniform nodal distributions are reported in Tables 3.3 and 3.4. Collapse load multiplier versus the number of nodes is also shown in Figure 3.7. Again, it can be observed that the iRBF-based method can provide more accurate solutions than the dRBF-based method, particularly for the static approach.

![Uniform nodal discretization](image)

Figure 3.6: Uniform nodal discretization

![Convergence of limit load factor for the plates](image)

Figure 3.7: Convergence of limit load factor for the plates

Table 3.5 shows that the results obtained by using RBF methods are in good agreement with previously reported numerical solutions. Considering upper bound limit factor, the present results are close to Zhou and Liu’s solutions, with the maximum error of only 2.79%. It is important to note that the estimated lower bounds reported in [90, 195] are higher than the present lower bound solutions, and surpasses the upper bound of the present iRBF method for the plate with square cutout. This can be explained by the fact that in [90, 195] the strong form
of the equilibrium equations was transformed into the so-called weak form, and to be satisfied locally in an average sense using approximated virtual displacement fields. Therefore, the static method in [90, 195] may result in a higher value than the actual limit multiplier. In contrast, it is clear that all the present lower bound solutions obtained are below the upper bounds reported in Table 3.5.

Table 3.3: Collapse multipliers for the square plate with a central square cutout

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>dRBF Upper bound</th>
<th>dRBF Lower bound</th>
<th>iRBF Upper bound</th>
<th>iRBF Lower bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>273</td>
<td>0.750</td>
<td>0.702</td>
<td>0.734</td>
<td>0.712</td>
</tr>
<tr>
<td>589</td>
<td>0.750</td>
<td>0.715</td>
<td>0.732</td>
<td>0.726</td>
</tr>
<tr>
<td>851</td>
<td>0.749</td>
<td>0.715</td>
<td>0.732</td>
<td>0.729</td>
</tr>
</tbody>
</table>

Table 3.4: Collapse multipliers for the square plate with a central thin crack

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>dRBF Upper bound</th>
<th>dRBF Lower bound</th>
<th>iRBF Upper bound</th>
<th>iRBF Lower bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>441</td>
<td>0.520</td>
<td>0.482</td>
<td>0.532</td>
<td>0.499</td>
</tr>
<tr>
<td>625</td>
<td>0.516</td>
<td>0.484</td>
<td>0.523</td>
<td>0.502</td>
</tr>
<tr>
<td>841</td>
<td>0.514</td>
<td>0.487</td>
<td>0.516</td>
<td>0.504</td>
</tr>
</tbody>
</table>

Table 3.5: Plates with cutouts problem: comparison with previous solutions

<table>
<thead>
<tr>
<th>Author</th>
<th>Approach</th>
<th>Square cutout</th>
<th>Thin crack</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present method, iRBF</td>
<td>Kinematic, static</td>
<td>0.732, 0.729</td>
<td>0.516, 0.504</td>
</tr>
<tr>
<td>Present method, dRBF</td>
<td>Kinematic, static</td>
<td>0.749, 0.715</td>
<td>0.514, 0.487</td>
</tr>
<tr>
<td>Pixin et al. [194], FEM</td>
<td>Kinematic</td>
<td>0.764 -</td>
<td>0.534 -</td>
</tr>
<tr>
<td>Zhou and Liu [88], NEM-Sibson</td>
<td>Kinematic</td>
<td>0.753 -</td>
<td>0.515 -</td>
</tr>
<tr>
<td>Zhou and Liu [88], NEM-Laplace</td>
<td>Kinematic</td>
<td>0.752 -</td>
<td>0.513 -</td>
</tr>
<tr>
<td>Belytschko and Hodge [193], FEM</td>
<td>Static</td>
<td>- 0.693 -</td>
<td>- 0.498 -</td>
</tr>
<tr>
<td>Zhang et al. [195], FEM</td>
<td>Static</td>
<td>- 0.747 -</td>
<td>- 0.514 -</td>
</tr>
<tr>
<td>Chen et al. [90], EFG</td>
<td>Static</td>
<td>- 0.736 -</td>
<td>- 0.513 -</td>
</tr>
</tbody>
</table>

3.3.3 Notched tensile specimen

Finally, a double notched specimen consists of a rectangular specimen with two thin cracks under in-plane tensile stresses $t_0$ as shown in Figure 3.8 (W = L = 2a = 1), is also considered. This problem exhibits volumetric locking phenomena
Chapter 3. Displacement and equilibrium mesh-free formulation based on iRBF for dual yield design

Figure 3.8: Double notch specimen

[196] and became a popular benchmark test for plastic yield design procedures. The locking problem was handled using various techniques proposed in the literature, including higher-order displacement-based finite element method [197], mixed finite elements [44, 63, 198] and discontinuous elements [59, 192, 199], mesh-free methods [95], smoothed finite elements [78, 79].

Figure 3.9: Convergence study for the double notched specimen problem

Owing to the symmetry, only the upper-right quarter of the double notched problem is discretized. Several uniform nodal distributions are employed. Computed solutions and convergence analysis are presented in Table 3.6 and Figure 3.9. Table 3.6 compares the present solutions with those obtained previously. The mean values of the dRBF and iRBF results are 1.1343 and 1.1342, respectively. It can be observed that these mean values are very close to the benchmark solution obtained using
mixed formulation by Christiansen and Andersen [198].

### 3.4 Conclusions

The present contribution has presented displacement and equilibrium mesh-free formulation based on integrated radial basis functions (iRBF) for dual yield design problems. In the kinematic formulation, the high-order approximation of the displacement fields using the integrated radial basis functions can prevent volumetric locking. Moreover, direct nodal integration of the iRBF approximation not only results in inexpensive computational cost, but also overcomes the instability problems. In the static formulation, with the use of iRBF approximation of the stress fields in combination with the collocation method, equilibrium equations and yield conditions only need to be enforced at the nodes, leading to the reduction in computational effort. It has been shown in several examples that the mean values of the iRBF upper and lower bounds are accurate, and can be considered as the actual collapse load multiplier for most practical engineering problems, for which exact solution is unknown.
Chapter 4

Limit state analysis of reinforced concrete slabs using an integrated radial basis function based mesh-free method

4.1 Introduction

This chapter presents an application of iRBF method for upper bound limit analysis of structures. This study aims to estimate the limit load as well as collapse mechanics of reinforced concrete slabs. Dealing with that structures, yield line or discontinuity layout optimization (DLO) methods can be employed. The element based yield line methods \cite{201,204} have the intrinsic advantage of providing accurate solutions for many practical engineering problems. However, the solutions of the element based yield line analysis are highly affected by the initial mesh topology because the yield-lines are restricted to be formed only at the edges of elements. Alternatively, discontinuity layout optimization, a generally applicable numerical limit analysis procedure that can be used to automatically identify the critical yield-line pattern, has been proposed in \cite{205,206}. However, owing to their advantages in treating problems of arbitrary geometries, complicated boundary conditions and complex loads, limit analysis procedures based on numerical discretization techniques has been found to be more popular \cite{58,64,93,207,209} in solving real-world engineering problems.

In the kinematic formulation, these unknown variables are often approximated in terms of nodal displacements and rotations. In order to minimize the total number of the problem degrees of freedom, and hence reduce computational effort, elements

without rotational degree of freedom, namely rotation-free elements, have been proposed by several researchers [210–213]. Taking advantages of such a rotation-free formulation, various rotation-free mesh-free based models have been proposed for thin plate structures [214–216]. Recently, a rotation-free formulation, that uses moving least squares approximation technique, has been developed for collapse analysis of reinforced concrete slabs [209]. It has been shown that the method can provide accurate collapse load multipliers with a relatively small number of degrees of freedom. Its main disadvantage, on the other hand, is the need to specially treat the kinematic boundary conditions due to the fact that the moving least squares approximation does not hold the so-called Kronecker delta property. Mesh-free method based on radial basis functions and point interpolation [217] may be used to overcome such the difficulty.

In this study, a novel rotational-free mesh-free formulation for limit state analysis of reinforced concrete slabs is developed. Note that the formulation for limit analysis of reinforced concrete slabs is very much different from those of plane problems, i.e., the formulation to determine the internal dissipation, the yield criterion used, and boundary conditions. The transverse velocity field is approximated by using the integrated radial basis functions (iRBF), particularly the multiquadric basis, and there is no rotational degree of freedom involved in the approximation. The resultant shape functions satisfy the Kronecker delta property, and hence displacement boundary conditions can be enforced in a way similar to one in the finite element method. The obtained discrete kinematic problem for limit state analysis of reinforced concrete slabs governed by Nielsen’s yield criterion is handled using available highly efficient solvers. Several reinforced concrete slabs of arbitrary geometries and different boundary conditions are examined, demonstrating that the proposed numerical procedure can provide accurate collapse load multipliers, and showing that yield-patterns in terms of plastic dissipation distribution can be automatically identified.

4.2 Kinematic formulation using the iRBF method for reinforced concrete slab

Consider a thin reinforced concrete slab of area $\Omega$, with kinematic boundary $\Gamma_u$. In the kinematic formulation, the approximation of the velocity field can be
expressed in terms of nodal velocities within the computational domain as follows

\[ u^h(x) = \sum_{I=1}^{N} \Phi_I(x) u_I \]  

(4.1)

The related rotations and curvatures are directly computed by differentiating the approximated velocity function as

\[ \theta^h_\alpha : = u^h_\alpha(x) = \sum_{I=1}^{N} \Phi_{I,\alpha}(x) u_I \]  

(4.2a)

\[ \kappa^h_{\alpha\beta} : = u^h_{\alpha\beta}(x) = \sum_{I=1}^{N} \Phi_{I,\alpha\beta}(x) u_I \]  

(4.2b)

where \( \Phi_I(x), \Phi_{I,\alpha}(x) \) and \( \Phi_{I,\alpha\beta}(x) \) are iRBF shape function and its derivatives described above.

For bending plate, the plastic dissipation function can be computed as

\[ D_p(\kappa) = \int_{\Omega} m^T \kappa d\Omega \]  

(4.3)

where \( m = [m_{xx} \ m_{yy} \ m_{xy}] \) presents moments on the yield surface associated with the plastic curvature rates \( \kappa \), which relates to the transverse velocity via the standard relations \( \kappa^T = [\kappa_{xx} \ \kappa_{yy} \ 2\kappa_{xy}] = \nabla^2 u \), in which the differential operator \( \nabla^2 \) is defined as

\[ \nabla^2 = \begin{bmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial y^2} & 2 \frac{\partial^2}{\partial x \partial y} \end{bmatrix} \]  

(4.4)

By means of numerical nodal integration, the plastic dissipation function can be expressed as

\[ D_p(\kappa) = \sum_{j=1}^{N} a_j (m_{px}^+ \kappa_x^+ + m_{py}^+ \kappa_y^+ + m_{px}^- \kappa_x^- + m_{py}^- \kappa_y^-) \]  

(4.5)

where \( a_j \) is the area of nodal representative domain \( j \), i.e., a Voronoi cell; \( (\kappa^+, \kappa^-) \) are the related curvatures; \( (m_{px}^+, m_{py}^-) \) present the positive and negative yield moments per unit length in \( x- \) and \( y- \) directions, which can be calculated as follows

\[ m_p = A_s f_Y d \left(1 - \frac{\phi}{2}\right) \]  

(4.6)
where $A_s$ and $f_Y$ denote the area and yield strength of reinforcement; dimension of $d$ is illustrated in Figure 4.1 and the reinforcement degree $\phi$ is given by

$$\phi = \frac{A_s f_Y}{d f_c}$$  \hspace{1cm} (4.7)

where $f_c$ is compressive strength of concrete.

For reinforced concrete slabs, the commonly used yield criterion is the Nielsen’s one, which can be expressed by two rotated quadratic cones

$$b_i + Q_i m \in \mathbb{R}^3, \quad i = 1, 2$$ \hspace{1cm} (4.8)

where

$$Q_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \quad Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$ \hspace{1cm} (4.9)

and

$$b_1^T = [m_{px}^+, m_{py}^+, 0] \quad b_2^T = [m_{px}^-, m_{py}^-, 0]$$ \hspace{1cm} (4.10)

Finally, the upper bound limit analysis of the reinforced concrete slabs can be formulated in the form of a conic optimization problem as follows conic optimization problem as follows
\[ \lambda^+ = \min \sum_{j=1}^{N} a_j (m_{px}^+ \kappa_x^+ + m_{py}^+ \kappa_y^+ + m_{px}^- \kappa_x^- + m_{py}^- \kappa_y^-)_j \] (4.11)

\[
\begin{align*}
\sum_{j=1}^{N} \Phi_{I,xx}(x_j)u_I &= (\kappa_x^- - \kappa_x^+)_j, \quad \forall j \in \{1, 2, \ldots, N\} \\
\sum_{j=1}^{N} \Phi_{I,yy}(x_j)u_I &= (\kappa_y^- - \kappa_y^+)_j, \quad \forall j \in \{1, 2, \ldots, N\} \\
\sum_{j=1}^{N} \Phi_{I,xy}(x_j)u_I &= \sqrt{2}(\kappa_{xy}^+ + \kappa_{xy}^-)_j, \quad \forall j \in \{1, 2, \ldots, N\}
\end{align*}
\] (4.12)

where \(A\) and \(b\) are obtained by imposing the unitary external work and the kinematic boundary conditions, respectively and given by

\[
A_{eq} = \begin{bmatrix}
\sum_{j=1}^{N} a_j \phi_1(x_j) & \sum_{j=1}^{N} a_j \phi_2(x_j) & \cdots & \sum_{j=1}^{N} a_j \phi_N(x_j) \\
\phi_1(x_1^b) & \phi_2(x_1^b) & \cdots & \phi_N(x_1^b) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_1(x_d^b) & \phi_2(x_d^b) & \cdots & \phi_N(x_d^b) \\
\phi_{1,x}(x_1^b) & \phi_{2,x}(x_1^b) & \cdots & \phi_{N,x}(x_1^b) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{1,x}(x_r^b) & \phi_{2,x}(x_r^b) & \cdots & \phi_{N,x}(x_r^b) \\
\phi_{1,y}(x_1^b) & \phi_{2,y}(x_1^b) & \cdots & \phi_{N,y}(x_1^b) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{1,y}(x_r^b) & \phi_{2,y}(x_r^b) & \cdots & \phi_{N,y}(x_r^b)
\end{bmatrix}
\] (4.13)

\[b_{eq} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}
\] (4.14)

where \((d, r_x, r_y)\) are number of boundary nodes having velocity, \(x\) and \(y\)-rotation conditions, respectively.

As above-presented, the rotations and curvatures are directly determined from the approximated transverse velocity. As a result, there is only one variable at each
node, and hence the size of the resultant optimization problem, (4.12), is kept to be minimum. Moreover, all equations in problem (4.12) are straightforwardly satisfied at discretized nodes within the computational domain. Problem (4.12) consists of \( N \) variables for nodal velocity and \( 3 \times N \) variables for each rotated cone. Therefore, the total variables in problem (4.12) is \( N_{\text{var}} = N + 2 \times 3 \times N = 7 \times N \).

The numerical implementation of the upper bound limit approach is a part of flow chart shown in Figure 1.1.

### 4.3 Numerical examples

This section investigates a number of benchmark problems, for which analytical and/or numerical solutions are available for comparison. For all examples, input data are: thickness \( t = 1 \); unit plastic moment of resistance \( m_p = 1 \); and slabs are subjected to the unit uniform pressure load \( q = 1 \). The solutions are obtained using Mosek optimization solver version 6.0 on a 2.8 GHz Intel Core i5 PC running Window 7.

#### 4.3.1 Rectangular slabs

Rectangular slabs with either simply supported (SSSS) or clamped (CCCC), (·) corresponds to left, bottom, right and top edges respectively, boundary conditions on all edges are considered first. It is assumed that the slabs are isotropic with positive and negative yield moments(\( m_p^+ = m_p^- = m_p \)) in both directions. Due to the symmetry, only the upper-right quarter of plate is modeled, as shown in Figure 4.2.

The influence of the shape parameter \( \alpha_s \) on the limit load factor of a simply supported square slab is studied first. The relationship between the computed limit load factors and the parameter \( \alpha_s \), is illustrated in Figure 4.3. It can be seen that for all nodal distribution solutions obtained when setting \( \alpha = 2 \) are lower (better) than those of smaller \( \alpha_s \), \( \alpha_s = 0.00001 \) or \( \alpha_s = 1 \). Note that when \( \alpha_s \) is taken to be larger than 2, a lower (i.e., improved) computed limit load factor may sometimes be obtained, however the computational cost increases. Therefore, in order to compromise between accuracy and computational cost \( \alpha_s \) is taken as 2 for all problems considered henceforth.
Various ratios of $b/a$ are investigated, and Table 4.1 summarizes computed numerical solutions using a regular nodal distribution of $35 \times 35$ nodes, corresponding to 8575 variables in the resultant optimization problem. Analytical solutions for a fully simply supported boundary condition slab are given as

$$\lambda = \begin{cases} 
\frac{24}{\sqrt{3 + \left(\frac{a}{b}\right)^2 - \frac{a}{b}}} \times \frac{m_p}{q_{ab}}, & \text{Ingerslev} \ [218]; \\
8 \left[1 + \frac{a}{b} + \frac{b}{a}\right] \times \frac{m_p}{q_{ab}}, & \text{Johansen} \ [219].
\end{cases} \tag{4.15}
$$

Table 4.1: Rectangular slabs with various ratios $b/a$: limit load factors

<table>
<thead>
<tr>
<th>$\frac{b}{a}$</th>
<th>Present method</th>
<th>Reference [209]</th>
<th>SSSS-Reference [218]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SSSS</td>
<td>CCC</td>
<td>SSSS</td>
</tr>
<tr>
<td>1.0</td>
<td>24.00</td>
<td>44.83</td>
<td>24.18</td>
</tr>
<tr>
<td>1.5</td>
<td>25.62</td>
<td>50.00</td>
<td>25.92</td>
</tr>
<tr>
<td>2.0</td>
<td>28.48</td>
<td>56.13</td>
<td>28.45</td>
</tr>
<tr>
<td>2.5</td>
<td>31.86</td>
<td>63.46</td>
<td>32.00</td>
</tr>
<tr>
<td>3.0</td>
<td>35.53</td>
<td>71.30</td>
<td>35.82</td>
</tr>
<tr>
<td>3.5</td>
<td>39.48</td>
<td>81.76</td>
<td>39.07</td>
</tr>
</tbody>
</table>
Chapter 4

Limit state analysis of reinforced concrete slabs using iRBF based mesh-free method

Figure 4.3: Simply supported square slab: normalized limit load factor \( \lambda^+ \) versus the parameter \( \alpha_s \)

Figure 4.4: Limit load factors \( \lambda^+ (m_p/q_a b) \) of rectangular slabs \( (b/a = 2) \) with different boundary conditions: CCCC (56.13), CCCF (48.53), CFCF (36.01), SSSS (28.48), FCCC (21.61), FCFC (9.08)

Rectangular slabs with other boundary conditions including free (F), simply supported (S) and clamped (C) edges are also considered. Limit load factors and convergence analysis for the case when \( b/a = 2 \) are illustrated in Figure 4.4.

To illustrate the performance of the iRBF based limit state analysis procedure,
the present solutions and associated computational aspects, including relative errors and number of variables, for simply supported and clamped square plates \((a = b = L)\) are compared with those obtained using the CS-HCT \[208\] and EFG methods \[209\], see Table 4.2. It can be observed that the proposed method can provide more accurate solutions with less computational effort (in terms of number of variables) than other selected approaches, particularly for simply supported slab. For simply supported slab, a solution of \(24.00m_p/qL^2\) is obtained by using the iRBF method with 8575 variables, which is better than results of \(24.07m_p/qL^2\) obtained by CS-HCT method \[208\] using 136188 variables, and of \(24.18m_p/qL^2\) presented in \[209\] using 8575 variables in the EFG formulation.

In Table 4.3, the iRBF solutions are also compared with previously published upper and lower bounds using displacement discontinuous finite elements \[64\], equilibrium mesh-free method \[93\] and equilibrium finite elements \[58, 207\]. It is evident
Table 4.3: Square slabs: limit load multipliers in comparison with other methods

<table>
<thead>
<tr>
<th>Author</th>
<th>Simple supported</th>
<th>Clamped</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda^+$</td>
<td>$\lambda^-$</td>
</tr>
<tr>
<td>Present method</td>
<td>24.00</td>
<td>-</td>
</tr>
<tr>
<td>Le et al. [208], CS-HCT</td>
<td>24.07</td>
<td>-</td>
</tr>
<tr>
<td>Le et al. [209], EFG</td>
<td>24.14</td>
<td>-</td>
</tr>
<tr>
<td>Bleyer et al. [64], FEM-T6b</td>
<td>24.00</td>
<td>-</td>
</tr>
<tr>
<td>Bleyer et al. [64], FEM-H3</td>
<td>24.43</td>
<td>-</td>
</tr>
<tr>
<td>Le et al. [93], FEM</td>
<td>-</td>
<td>23.96</td>
</tr>
<tr>
<td>Krabbenhoft [58], FEM</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Maunder et al. [207], FEM</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

that these solutions are, in general, in good agreement. Note that in approaches proposed by others, there is at least 3 variables per node, and hence the size of corresponding formulation may be larger than that of the present method. The yield patterns in terms of plastic dissipation distribution for rectangular slabs with different boundary conditions are also plotted in Figure 4.5.

4.3.2 Regular polygonal slabs

Next, regular polygonal slabs with $n$-sides ($n = 3, 4, 5, 6, +\infty$) are examined. Nodal distribution and computational domains of all slabs are shown in Figure 4.6. For square and circular slabs, only the upper-right quarters are modeled, while for triangular, pentagonal and hexagonal slabs the whole domains are discretized. All slabs are assumed to be isotropic with equal magnitudes of hogging and sagging yield moments $m_p^+ = m_p^- = m_p$ in both directions. Let $R$ denotes the radius of incircle of regular polygons. Both simply supported and clamped boundary conditions are investigated.

Computed limit load multipliers and associated dissipation distribution for polygonal slabs are reported in Figure 4.7. Table 4.4 compares the present results for clamped slabs with analytical solutions and recent selected upper bounds obtained using CS-HCT and EFG based numerical procedures. For all cases, the EFG based approach can provide lower (better) upper bound solutions than the present iRBF method. This may be explained by the fact that the plastic dissipation along clamped boundaries can be accurately produced by the high-order shape functions obtained by the moving least squares approximation technique, that uses
Chapter 4. Limit state analysis of reinforced concrete slabs using iRBF based mesh-free method

Figure 4.6: Nodal distribution and computational domains of polygonal slabs: (a) triangle; (b) square; (c) pentagon; (d) hexagon; (e) circle

an isotropic quartic spline weight function. However, the advantages of the iRBF method over the EFG approach are that the iRBF shape functions hold the Kronecker delta property, and hence there is no need of any special treatment when enforcing boundary conditions as encountered in the EFG method; and that the computation of iRBF shape functions is less expensive in terms of CPU time than that of EFG’s ones, i.e., for a mesh of $15 \times 15$ nodes, the iRBF method takes approximately 0.5s, compared with about 5.8s when using the EFG method.

Table 4.4: Clamped regular polygonal slabs: limit load factors in comparison with other solutions ($m_p/qR^2$)

<table>
<thead>
<tr>
<th>Author</th>
<th>Geometry of slabs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present method</td>
<td>Triangle</td>
</tr>
<tr>
<td></td>
<td>10.42</td>
</tr>
<tr>
<td>Le et al. [208], CS-HCT</td>
<td>10.67</td>
</tr>
<tr>
<td>Le et al. [209], EFG</td>
<td>9.98</td>
</tr>
<tr>
<td>Fox [220], analytical method</td>
<td>9.61</td>
</tr>
<tr>
<td>Johansen [219], analytical method</td>
<td>-</td>
</tr>
</tbody>
</table>

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4.3.3 Arbitrary geometric slab with a rectangular hole

The last example comprises an arbitrary geometric slab with an eccentric rectangular cutout, as shown in Figure 4.8, which has been examined previously using equilibrium finite elements [200], curvature smoothing HCT elements [208] and the displacement mesh-free method [209]. The problem is solved using a nodal distribution of 1151 nodes, corresponding to 8051 variables in the resultant optimization problem.

Figure 4.7: Plastic dissipation distribution and collapse load multipliers \((m_p/qR^2)\) of polygonal slabs: \((a, b, c, d, e)\)-clamped; \((f, g, h, i, j)\)-simply supported

Figure 4.8: Arbitrary shape slabs: geometry (all dimensions are in meter) and discretization
Chapter 4 Limit state analysis of reinforced concrete slabs using iRBF based mesh-free method

For isotropic slab with equal positive and negative yield moments \((m_p^+ = m_p^- = m_p)\), the collapse load factor obtained is \(0.1421 \times m_p\), which is in good agreement with results of \(0.1480 \times m_p\) in [201], of \(0.1420 \times m_p\) in [208] and of \(0.1424 \times m_p\) in [209]. Table 4.5 summarizes limit load multipliers of slabs with various ratios of \(m_p^+ / m_p^-\), illustrating the influence of negative yield moment on the bearing capacity of slabs. The present results are competitive with those reported in [208, 209]. Moreover, it is interesting to point out that, here, for all cases involving simply supported boundary conditions the present iRBF method can result in lower (better) solutions than the EFG approach. Orthotropic slab with the ratio of yield moments in \(x-\) and \(y-\)directions \(m_{px}/m_{py} = 0.5\) is also considered. The computed result of \(0.086 \times m_p\) is in excellent agreement with a solution of \(0.086 \times m_p\) reported in [208, 209]. In general, the present method is more efficient than those of [208, 209].

Plastic dissipation distribution and collapse mechanism for the case of isotropic reinforcement are also shown in Figure 4.9. It can be observed that the failure mechanism obtained by present method is in good agreement compared with one in [205] using DLO approach.

![Displacement contour](image1.png)  ![Dissipation distribution](image2.png)  ![Failure mechanism](image3.png)

Figure 4.9: Arbitrary geometric slab with an eccentric rectangular cutout \((m_p^+ = m_p^- = m_p)\): displacement contour and dissipation distribution at collapse state
4.4 Conclusions

A novel rotation-free mesh-free method based on integrated radial basis functions has been developed for limit state analysis of reinforced concrete slabs. The transverse velocity is approximated without using rotational degrees of freedom, and therefore the total number of variables in the resultant optimization problem is kept to a minimum, i.e., equal to the number of discretized nodes in the problem domain. The proposed formulation is tested by applying it to various Nielsen’s reinforced concrete slabs of arbitrary geometries. It has been demonstrated that the present method, consisting of high-order shape functions obtained by integrating radial basis functions, can provide accurate collapse load multipliers. Moreover, the high-order and smooth iRBF approximation is capable of capturing yield patterns of arbitrary geometric slabs. The present optimization strategy based on conic programming enables solutions of practical sized reinforced concrete slabs to be obtained rapidly. It should be noted that in the mesh-free based numerical procedures for limit state analysis of structures nodes may be moved, discarded or introduced conveniently. Hence the implementation of an h-adaptive scheme is facilitated, and will be the subject of future research.

<table>
<thead>
<tr>
<th>$m_p^+$ ($m_p^-$</th>
<th>1</th>
<th>1/2</th>
<th>1/4</th>
<th>1/8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present method ($N_{var} = 8057$)</td>
<td>0.1421</td>
<td>0.1295</td>
<td>0.1217</td>
<td>0.1167</td>
</tr>
<tr>
<td>Le et al. [208], CS-HCT ($N_{var} = 38139$)</td>
<td>0.1420</td>
<td>0.1298</td>
<td>0.1233</td>
<td>0.1217</td>
</tr>
<tr>
<td>Le et al. [209], EFG ($N_{var} = 8057$)</td>
<td>0.1424</td>
<td>0.1299</td>
<td>0.1226</td>
<td>0.1181</td>
</tr>
</tbody>
</table>
Chapter 5

A stabilized iRBF mesh-free method for quasi-lower bound shakedown analysis of structures

5.1 Introduction

In chapters 3 and 4 the iRBF-based mesh-free method is extent to the limit analysis of structures, where it is assumed that loading increases gradually until the collapse appears. In fact, engineering structures are usually subjected to repeat, cyclic or time-dependent loads. Under a repeated cycle of loading, the structures may be fail due to some collapse modes, e.g. rotating plasticity, a general mode of alternating plasticity (lower cycle fatigue) [32], incremental plasticity (ratcheting) or instantaneous plasticity. Direct analysis, a perfect alternative scheme for step-by-step method, has been successfully applied for this field. Limit analysis for the case of proportional loading and shakedown analysis in case of variable repeated loading, have been found to be more efficient [19, 73, 77, 221–223]. In direct shakedown analysis, the load limits can be determined without a need of loading history, and hence the method can be applied to a wide range of problems in engineering practices.

The implementation of computational shakedown analysis generally involves two main steps: (i) approximate the problem fields using a discretization method, and (ii) solve the resulting optimization to obtain the solution. In the literature, various numerical approaches have been developed for both kinematic and static shakedown analysis, for instance, mesh-based methods [67, 80, 82, 224, 226], boundary element method [86, 87] and mesh-free approaches [90, 91]. When the variable fields

are approximated and bound theorems is applied, shakedown analysis becomes an optimization problem which can be solved using iterative algorithm \[86, 90\] or primal-dual interior-point method \[73, 80\]. However, in treating large-scale optimization problem, the second-order cone programming has been proved to be more powerful \[32, 77\].

The objective of this study is to extend the iRBF mesh-free method to quasi-static shakedown analysis of 2D and 3D structures. In the quasi-static formulation, the stress field is decomposed into two parts involving a fictitious elastic stress and a self-equilibrated residual stress. The fictitious elastic stresses are calculated using the usual Galerkin procedure. The virtual strains are approximated by the (stabilized) iRBF shape functions, and equilibrium equations for the self-equilibrated residual stress field are enforced in a weak form. It is worth noting that the present formulation is different from the one presented in chapter 3, where total stress fields are approximated and a strong form of equilibrium equations are used. The yield conditions for two and three dimensional problems are formulated as conic constraints. All constrains of the resulting optimization problem are imposed at a finite number of discretized nodes, instead of Gaussian points, and hence the size of the obtained optimization problem is kept to a minimum. The combination of iRBF mesh-free method and conic programming enables the shakedown solutions to be obtained rapidly, and consequently the load domains consisting of a large number of points can be approximated efficiently. The performance of proposed procedure will be illustrated by investigating various benchmark problems in plane stress, plane strain and three-dimensions conditions.

### 5.2 iRBF discretization for static shakedown formulation

Consider an elastic-perfectly plastic structure of volume \(V\) subjected to variable repeated loads. Let \(\sigma^E\) denote the fictitious elastic stress belonging to a bounded time-independent global loading domain \(\mathcal{P} = \{ \sigma^E | \sigma^E(x, t) \in \mathcal{P}_x, x \in V, t \in [0, T] \}\), where \(\mathcal{P}_x\) is the local loading domain at a point \(x \in V\). The static/lower bound shakedown theorem states that if there exits a residual stress \(\rho\), which is time-independent and self-equilibrium, so that the total stress, \(\sigma = \sigma^E + \rho\), does not violate the yield condition at any point in the structure for all possible load combination. Let \(\lambda\) be the shakedown safety factor, the lower bound on the actual shakedown safety factor of a structure, \(\lambda_s\), can be determined by solving the
Following optimization problem

$$\lambda_s = \max_{\mathbf{\rho} \in \mathcal{R}} \{ \lambda \mid \psi(\lambda \mathbf{\sigma}^E + \mathbf{\rho}, \sigma_p) \leq 0, \forall \mathbf{\sigma}^E \in \mathcal{R} \}$$  (5.1)  

where $\mathcal{R}$ is the set of admissible bounded residual stress field, $\sigma_p$ is the yield stress, and $\psi$ is the yield function of ductile materials.

In terms of numerical implementation, the fictitious residual stress field can be approximated via a reflection of nodal values in the problem domain using the iRBF method as follows

$$\mathbf{\rho}^h(x) = \sum_{i=1}^{N} \Phi_i(x) \rho_i$$  (5.2)  

where $\Phi_i(x)$ is the iRBF shape function; the residual stresses at nodes are denoted by a vector consisting $(\rho_{xx}, \rho_{yy}, \rho_{xy})$ for 2D and $(\rho_{xx}, \rho_{yy}, \rho_{zz}, \rho_{xy}, \rho_{xz}, \rho_{yz})$ for 3D discretizations.

In the equilibrium shakedown analysis formulation, the residual stress fields is required to be equilibrated at every point in the problem domain. This results in difficulties in a numerical solution strategy due to the fact that equilibrium equations are often accessed at integration points. A way out of such the difficulties is to transform the strong form of the equilibrium equations into its weak form by using the principle of virtual work as follows

$$\int_V \delta \mathbf{\epsilon}^T(x) \mathbf{\rho}(x) dV = 0$$  (5.3)  

where $\delta \mathbf{\epsilon}(x)$ denotes any virtual strain which satisfies the kinematic boundary conditions. The virtual strain field $\delta \mathbf{\epsilon}(x)$ can be approximated using the iRBF method as

$$\delta \mathbf{\epsilon}(x) = \mathbf{B}(x) \delta \mathbf{d}$$  (5.4)  

where $\delta \mathbf{d}$ denotes the nodal displacement vector and $\mathbf{B}(x)$ is the strain-displacement matrices defined for 2D problems as

$$\mathbf{B}(x) = \begin{bmatrix} \phi_{1,x} & 0 & \phi_{2,x} & 0 & \cdots & \phi_{N,x} & 0 \\ 0 & \phi_{1,y} & 0 & \phi_{2,y} & \cdots & 0 & \phi_{N,y} \\ \phi_{1,y} & \phi_{1,x} & \phi_{2,y} & \phi_{2,x} & \cdots & \phi_{N,y} & \phi_{N,x} \end{bmatrix}$$  (5.5)  

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and for 3D problems as

\[
B(x) = \begin{bmatrix}
\phi_{1,x} & 0 & \phi_{2,x} & 0 & \cdots & \phi_{N,x} & 0 & 0 \\
0 & \phi_{1,y} & 0 & \phi_{2,y} & 0 & \cdots & 0 & \phi_{N,y} & 0 \\
0 & 0 & \phi_{1,z} & 0 & \phi_{2,z} & \cdots & 0 & 0 & \phi_{N,z} \\
\phi_{1,y} & \phi_{1,x} & 0 & \phi_{2,y} & \phi_{2,x} & \cdots & 0 & \phi_{N,y} & \phi_{N,x} \\
0 & \phi_{1,z} & \phi_{1,y} & 0 & \phi_{2,z} & \phi_{2,y} & \cdots & 0 & \phi_{N,z} \\
\phi_{1,z} & 0 & \phi_{1,x} & \phi_{2,z} & 0 & \phi_{2,x} & \cdots & \phi_{N,z} & 0 & \phi_{N,x}
\end{bmatrix}
\] (5.6)

With the use of the iRBF approximated virtual strain field, the weak form (5.3) can be rewritten as

\[
\int_{\mathcal{V}} [B(x)\delta d]^T \rho(x) d\mathcal{V} = \delta d^T \int_{\mathcal{V}} B^T(x) \rho(x) d\mathcal{V} = 0
\]

Equation (5.7) must hold for all \(\delta d\), hence one can obtain

\[
\int_{\mathcal{V}} B^T(x) \rho(x) d\mathcal{V} = \sum_{k=1}^{N} \mathcal{V}_k B_k^T \rho_k = C_{eq} \rho = 0
\]

where \(\mathcal{V}_k\) is the volume of a representative Voronoi domain of node \(k\), \(C_{eq}\) is a constant equilibrium matrix, in which the essential boundary conditions are taken into account by eliminating the corresponding degrees of freedom of nodes on the kinematic boundaries. Therefore, the number of rows of \(C_{eq}\) is reduced to \((sdof - N_{BC})\), where \(sdof\) denotes the number of total degrees of freedom of the system, and \(N_{BC}\) is the number of degrees of freedom of nodes on the kinematic boundaries.

In the next step, the mathematical algorithms will be extent to handle the optimization problem. Using the primal-dual interior-point algorithms, the von Mises criterion will be employed and formulated for 2D problems as follows

\[
\psi(\sigma) = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx} \sigma_{yy} + 3\sigma_{xy}^2} - \sigma_y \quad \text{for plane stress} \quad (5.9a)
\]

\[
\psi(\sigma) = \sqrt{\frac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2} - \sigma_y \quad \text{for plane strain} \quad (5.9b)
\]

and for 3D problems as

\[
\psi(\sigma) = (\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{xx} - \sigma_{zz})^2 + (\sigma_{yy} - \sigma_{zz})^2 + 6(\sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2) - 2\sigma_p^2 \quad (5.10)
\]
Introducing auxiliary variables \( \mathbf{r} \) defined by

\[
\begin{align*}
    r_1 &= \sigma_p; \quad r_{2 \rightarrow 4} = J_1 \sigma = J_1 (\lambda \sigma^e + \rho) \quad \text{for plane stress} \quad (5.11a) \\
    r_1 &= \sigma_p; \quad r_{2 \rightarrow 3} = J_2 \sigma = J_2 (\lambda \sigma^e + \rho) \quad \text{for plane strain} \quad (5.11b) \\
    r_1 &= \sqrt{2} \sigma_p; \quad r_{2 \rightarrow 7} = J_3 \sigma = J_3 (\lambda \sigma^e + \rho) \quad \text{for 3D} \quad (5.11c)
\end{align*}
\]

where

\[
\begin{align*}
    J_1 &= \frac{1}{2} \begin{bmatrix} 2 & -1 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{3} \end{bmatrix}; \quad J_2 = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (5.12)
\end{align*}
\]

and

\[
J_3 = \frac{1}{\sqrt{2} \sigma_0} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \end{bmatrix} \quad (5.13)
\]

The von Mises failure criterion \( \psi(\lambda \sigma^e + \rho) \) can be now rewritten in terms of standard conic constraints as

\[
\begin{align*}
    \mathcal{L} &= \left\{ \mathbf{r} \in \mathbb{R}^4 \mid r_1 \geq \|r_{2 \rightarrow 4}\|_{L^2} = \sqrt{r_2^2 + r_3^2 + r_4^2} \right\} \quad \text{for plane stress} \quad (5.14a) \\
    \mathcal{L} &= \left\{ \mathbf{r} \in \mathbb{R}^3 \mid r_1 \geq \|r_{2 \rightarrow 3}\|_{L^2} = \sqrt{r_2^2 + r_3^2} \right\} \quad \text{for plane strain} \quad (5.14b) \\
    \mathcal{L} &= \left\{ \mathbf{r} \in \mathbb{R}^6 \mid r_1 \geq \|r_{2 \rightarrow 7}\|_{L^2} = \sqrt{r_2^2 + r_3^2 + r_4^2 + r_5^2 + r_6^2} \right\} \quad \text{for 3D} \quad (5.14c)
\end{align*}
\]

Finally, the equilibrium formulation of a direct analysis problem can be expressed as follows

\[
\begin{align*}
    \lambda_s &= \max \lambda \\
    \text{s.t.} \quad & \begin{cases} 
        C_{eq} \rho = 0 \\
        r_{kt} \in \mathcal{L}_{kt}, \quad k = 1, 2, \ldots, N; \quad t = 1, 2, \ldots, M
    \end{cases} \quad (5.15)
\end{align*}
\]

where \( \mathbf{r}_{kt} \) is the additional vector defined at the discretized nodes \( k^{th} \) for the loading.
vertex \( t^{th} \), \( M = 2^{n_L} \) is the number of vertices of the convex polyhedral load domain, in which \( n_L \) is the number of independent loading processes. It is important to note that the safety load multipliers \( \lambda_s \) obtained from problem (5.15) are quasi-lower-bound on the actual solutions. This is because the fact that equilibrium equations and yield condition in the present formulation are enforced and satisfied at a finite number of nodes in the computational domain.

The whole numerical implementation of quasi-lower bound limit and shakedown formulation for structural analysis is summarized in Figure 5.1.
5.3 Numerical examples

In this section, various examples in two-and three-dimensions are examined to illustrate the performance of proposed method. The resulting optimization problems are solved using the commercial software package Mosek on a 2.8 GHz Intel Core i7 PC running Window 10. The number of variables $N_{\text{var}}$ in the resulting optimization problem is equal to $3 \times N + 1 + 4 \times 2^n L$ for plane stress, $3 \times N + 1 + 3 \times 2^n L$ for plane strain and $6 \times N + 1 + 6 \times 2^n L$ for 3D problems. For comparison purpose, numerical solutions based on the radial point interpolation method (RPIM) are also reported.

5.3.1 Punch problem under proportional load

In order to study computational aspects of the present iRBF-based quasi-static direct analysis procedure, the punch problem consisting of a semi-infinite rigid-plastic von Mises medium under a punch load of $2\tau_0$ ($n_L = 1$) is considered. Note that the problem has been investigated in [227] using both kinematic and static formulation based on the iRBF mesh-free method. The problem is solved using various nodal distribution in the computational domain of $B = 5$, $H = 2$ and $a = 1$ as shown in Figure 5.3.

![Figure 5.2: Prandtl’s punch problem](image)

Figure 5.2 compares the present computed solutions with results using the iRBF static method presented in [227]. It can be observed that present solutions converge from above while results in [227] approach to the exact collapse load factor from below. Both methods are based on equilibrium formulation, but convergence behaviour is different. This may be explained by the fact that in the present formulation the virtual displacement fields are approximated and equilibrium equations are satisfied in a weak form, whereas in [227] stress fields are approximated and the strong form of equilibrium conditions are enforced using collocation technique.
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Figure 5.3: Prandtl’s punch problem: computational model

Figure 5.4: The punch problem: computational analysis

It is also interesting to point out that both methods results in the same number of variables in its optimization problem when using the same nodal distribution, but the present method can provide more accurate solutions (smaller relative errors).

Table 5.1: Computational results of iRBF and RPIM methods

<table>
<thead>
<tr>
<th>Approach</th>
<th>Collapse load factor</th>
<th>Relative error (%)</th>
<th>Optimization time (s)</th>
<th>Exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>iRBF, quasi lower bound</td>
<td>5.186</td>
<td>0.86</td>
<td>79.34</td>
<td>12355</td>
</tr>
<tr>
<td>RPIM, quasi lower bound</td>
<td>5.208</td>
<td>1.29</td>
<td>122.72</td>
<td>12355</td>
</tr>
<tr>
<td>RPIM, lower bound in [227]</td>
<td>5.087</td>
<td>1.07</td>
<td>214.41</td>
<td>12355</td>
</tr>
</tbody>
</table>

$t$ is the CPU optimization time, $e$ is the relative errors in collapse load factor

Table 5.1 reports computed collapse load factors, relative errors, computational
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![Graph (a)](image1.png)  ![Graph (b)](image2.png)

(a) Relative errors  
(b) Convergence rate

Figure 5.5: The punch problem: iRBF versus RPIM

![Image (a)](image3.png)  ![Image (b)](image4.png)  ![Image (c)](image5.png)

(a) Elastic stress field  
(b) Residual stress field  
(c) Limit stress field

Figure 5.6: Prandtl’s punch problem: distribution of elastic, residual and limit stress fields

CPU time using both iRBF and RPIM based numerical procedures with a nodal distribution of 2059 nodes (using 2D model). Convergence analysis of the two methods is also illustrated in Figure 5.5. It can be observed that for all nodal distribution the iRBF method results in more accurate collapse load factors than the RPIM method, while the computational CPU time taken to solve the iRBF optimization problem is smaller that that of the RPIM method. In short, the present method is more advantaged than the RPIM approach and the iRBF based static method presented in [227] in terms of computational efficiency and solution accuracy. The distribution of elastic, residual stresses and stress field at limit state is also shown in Figure 5.6.
5.3.2 Thin plate with a central hole subjected to variable tension loads

Next, consider a square plate with a circular hole at its center, see Figure 5.7, and subjected to a biaxial tension loads varying independently as

\[ 0 \leq p_1 \leq p_{01}, \quad 0 \leq p_2 \leq p_{02} \]  \tag{5.16}

Figure 5.7: Square plate with a central circular hole: geometry (thickness \( t = 0.4R \)), loading and computational domain

The problem has been extensively investigated in the literature using different numerical procedures, for example static formulation [82, 90, 193, 197, 221, 224, 225], kinematic formulation [68, 78, 80, 96, 191], mixed formulation [226]. The plate is solved employing only upper-right quarter, see 5.7, and using the following data: \( E = 2.1 \times 10^5 \) MPa, \( \nu = 0.3 \) and \( \sigma_p = 200 \) MPa. The two- and three-dimension nodal discretization and associated Voronoi diagrams are respectively plotted in Figures 5.8(a) and 5.8(b).

Tables 5.2 and 5.3 report computed limit and shakedown load factors using the proposed numerical procedure, together with those obtained using different methods in the literature [80, 86, 87, 90, 96, 226], for three different loading cases. In general, good agreement of these solutions is observed. However, the present method posses more advantages in terms of discretization technique and optimization algorithm, compared with previously proposed approaches. The iRBF method results in high-
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(a) 2D model  
(b) 3D model

Figure 5.8: Square plate with a central circular hole: the nodal distribution and Voronoi diagrams

(a) Load domains \(t = 0.4R\)  
(b) Limit load domains for various thickness

Figure 5.9: Plate with hole: loading domain

order shape functions than finite elements \([226]\), smoothed finite elements \([80]\), and boundary elements \([86, 87]\), and hence more accurate solutions can be obtained when using the same mesh. Note that in \([90]\), the radial point interpolation mesh-free method (RPIM) is used in the kinematic formulation, but as shown in the first example the method does not perform as well as the iRBF method. In \([90]\), the EFG mesh-free method is used in the framework of static theorem, providing accurate solutions. However, the EFG shape functions do not hold Kronecker’s delta properties, and hence attention must be paid to enforce boundary conditions.
Figure 5.10: Plate with hole: load domains in comparison with other numerical methods

This is not the case for the iRBF method, in which boundary conditions can be enforced in a way similar to ones in the finite element method. Regarding the optimization algorithm, the second-order cone programming used in the present numerical procedure is able to solve large-scale problems with up to thousands of variables in a couple of minutes, enabling the efficient computation of a large number of points to describe a load domain. Stress fields for various load cases are plotted in Figures 5.11–5.13. Limit load domains for various plate thickness are also shown in Figure 5.9(b).

Graphics of proportional plastic limit curve and shakedown limit curve for all range of (5.16) are plotted in the plane of load coordinates \( p_1/\sigma_p, p_2/\sigma_p \) as in Figure 5.9(a). Figure 5.10 shows the limit and shakedown load domains using the iRBF based static method and other numerical approaches. It is evident that iRBF solutions are in good agreement with those obtained using static theorem [90, 193, 221] and kinematic formulation [166, 228].
### Table 5.2: Plate with hole: comparison of limit load multipliers

<table>
<thead>
<tr>
<th>Authors</th>
<th>Loading cases</th>
<th>$p_1 = p_2$</th>
<th>$p_1 = 2p_2$</th>
<th>$p_2 = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present iRBF 2D, quasi-static</td>
<td>0.871</td>
<td>0.902</td>
<td>0.8001</td>
<td></td>
</tr>
<tr>
<td>Present iRBF 3D, quasi-static</td>
<td>0.858</td>
<td>0.909</td>
<td>0.8002</td>
<td></td>
</tr>
<tr>
<td>Ho et al. [82], CS-FEM, quasi-static</td>
<td>0.896</td>
<td>0.911</td>
<td>0.8007</td>
<td></td>
</tr>
<tr>
<td>Gross-Weege [221], FEM, static</td>
<td>0.882</td>
<td>0.891</td>
<td>0.792</td>
<td></td>
</tr>
<tr>
<td>Liu et al. [80], EFG, static</td>
<td>0.903</td>
<td>0.915</td>
<td>0.795</td>
<td></td>
</tr>
<tr>
<td>Chen et al. [90], EFG, static</td>
<td>0.874</td>
<td>0.899</td>
<td>0.798</td>
<td></td>
</tr>
<tr>
<td>Tin-Loi and Ngo [197], FEM, static</td>
<td>0.895</td>
<td>0.912</td>
<td>0.803</td>
<td></td>
</tr>
<tr>
<td>Vicente da Silva and Antao [191], FEM, kinematic</td>
<td>0.899</td>
<td>0.915</td>
<td>0.807</td>
<td></td>
</tr>
<tr>
<td>Le et al. [78], CS-FEM, kinematic</td>
<td>0.895</td>
<td>0.911</td>
<td>0.801</td>
<td></td>
</tr>
<tr>
<td>Zouain et al. [226], FEM, mixed</td>
<td>0.894</td>
<td>0.911</td>
<td>0.903</td>
<td></td>
</tr>
<tr>
<td>Gaydon and McCrum [230], analytical solution</td>
<td>-</td>
<td>-</td>
<td>0.800</td>
<td></td>
</tr>
</tbody>
</table>

### Table 5.3: Plate with hole: comparison of shakedown load multipliers

<table>
<thead>
<tr>
<th>Authors</th>
<th>Loading cases</th>
<th>$p_1 = p_2$</th>
<th>$p_1 = 2p_2$</th>
<th>$p_2 = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present iRBF 2D, quasi-static</td>
<td>0.478</td>
<td>0.551</td>
<td>0.650</td>
<td></td>
</tr>
<tr>
<td>Present iRBF 3D, quasi-static</td>
<td>0.474</td>
<td>0.546</td>
<td>0.645</td>
<td></td>
</tr>
<tr>
<td>Ho et al. [82], quasi-static</td>
<td>0.449</td>
<td>0.536</td>
<td>0.617</td>
<td></td>
</tr>
<tr>
<td>Gross-Weege [221], static</td>
<td>0.446</td>
<td>0.524</td>
<td>0.614</td>
<td></td>
</tr>
<tr>
<td>Genna [225], static</td>
<td>0.478</td>
<td>0.566</td>
<td>0.653</td>
<td></td>
</tr>
<tr>
<td>Liu et al. [87], static</td>
<td>0.477</td>
<td>0.549</td>
<td>0.647</td>
<td></td>
</tr>
<tr>
<td>Carvelli et al. [68], kinematic</td>
<td>0.518</td>
<td>0.607</td>
<td>0.696</td>
<td></td>
</tr>
<tr>
<td>Corradi and Zavelani [228], kinematic</td>
<td>0.504</td>
<td>0.579</td>
<td>0.654</td>
<td></td>
</tr>
<tr>
<td>Krabbenhoft [229], kinematic</td>
<td>0.430</td>
<td>0.499</td>
<td>0.595</td>
<td></td>
</tr>
<tr>
<td>Zouain et al. [226], mixed</td>
<td>0.429</td>
<td>0.500</td>
<td>0.594</td>
<td></td>
</tr>
<tr>
<td>Garcea et al. [231], mixed</td>
<td>0.438</td>
<td>0.508</td>
<td>0.604</td>
<td></td>
</tr>
<tr>
<td>Tran et al. [80], dual algorithms</td>
<td>0.444</td>
<td>0.514</td>
<td>0.610</td>
<td></td>
</tr>
</tbody>
</table>
5.3.3 Grooved plate subjected to tension and in-plane bending loads

A grooved plate subjected to in-plane tension load $p_N$ and bending load $p_M$ is also considered. The variable loads are defined by

$$0 \leq p_N \leq \sigma_p; \quad 0 \leq p_M \leq \sigma_p$$ (5.17)

Geometry, loading, boundary conditions and computational nodal distribution
are shown in Figure 5.15. Geometry and material properties are given as: $R = 250 \text{ mm}$, $L = 4R$, $E = 2.1 \times 10^5 \text{ MPa}$, $\nu = 0.3$, $\sigma_p = 116.2 \text{ MPa}$.

Figure 5.14: Grooved square plate subjected to tension and in-plane bending loads

Figure 5.15: Grooved square plate: computational nodal distribution

Limit and shakedown interaction diagram is illustrated in the plane of load co-ordinates $p_N/\sigma_p, p_M/\sigma_p$ as in Figure 5.14(b). Table 5.4 reports computed limit and shakedown load multipliers for two loading cases: (i) pure tension ($p_N \neq 0, p_M = 0$); (ii) both tension and bending ($p_N \neq 0, p_M \neq 0$). It is evident that the iRBF solutions agree well with published results using different discretization methods. Stress fields for various load cases are plotted in Figures 5.16 and 5.17.
### Table 5.4: Grooved plate: present solutions in comparison with other results

<table>
<thead>
<tr>
<th>Authors</th>
<th>Limit load factor</th>
<th>Shakedown load factor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( p_N \neq 0 )</td>
<td>( p_N \neq 0 )</td>
</tr>
<tr>
<td>Present iRBF 2D</td>
<td>0.524</td>
<td>0.25832</td>
</tr>
<tr>
<td>Present iRBF 3D</td>
<td>0.571</td>
<td>0.26418</td>
</tr>
<tr>
<td>Ho et al. [82], CS-FEM</td>
<td>0.557</td>
<td>0.29394</td>
</tr>
<tr>
<td>Tran et al. [80], ES-FEM</td>
<td>0.562</td>
<td>0.27811</td>
</tr>
<tr>
<td>Nguyen-Xuan et al. [81], ES-FEM</td>
<td>0.559</td>
<td>0.29660</td>
</tr>
<tr>
<td>Tran [232], ES-FEM</td>
<td>0.572</td>
<td>0.30498</td>
</tr>
<tr>
<td>Vu [233], FEM</td>
<td>0.557</td>
<td>-</td>
</tr>
<tr>
<td>Prager and Hodge [234], FEM</td>
<td>0.500</td>
<td>-</td>
</tr>
<tr>
<td>Casciaro [235], FEM</td>
<td>0.568</td>
<td>-</td>
</tr>
<tr>
<td>Yan [236], numerical</td>
<td>0.558</td>
<td>-</td>
</tr>
<tr>
<td>Yan [236], analytical</td>
<td>0.500 - 0.577</td>
<td>-</td>
</tr>
</tbody>
</table>

(a) Elastic stress field  
(b) Residual stress field  
(c) Limit stress field

Figure 5.16: Grooved plate: stress fields in case of \( [p_N, p_M] = [\sigma_p, 0] \)

(a) Elastic stress field  
(b) Residual stress field  
(c) Limit stress field

Figure 5.17: Grooved plate: stress fields in case of \( [p_N, p_M] = [\sigma_p, \sigma_p] \)
5.3.4 A symmetric continuous beam

This example deals with a symmetric continuous beam subjected to two independently variable loads \( p_1 \in [1.2, 2] \) and \( p_2 \in [0, 1] \), as presented in Figure 5.18(a). The material properties are assumed as: \( E = 1.8 \times 10^5 \) MPa, \( \nu = 0.3 \), \( \sigma_p = 100 \) MPa. Tables 5.5 and 5.6 compares the iRBF results with published solutions using finite elements [231], cell-based smoothed finite elements [82], node-based strain smoothing method [81] and the EFG mesh-free method [90].

![Figure 5.18: Symmetric continuous beam subjected to two independent load](image)

**Table 5.5: Symmetric continuous beam: limit load factors**

<table>
<thead>
<tr>
<th>Authors</th>
<th>( p_1 = 2.0 )</th>
<th>( p_1 = 0.0 )</th>
<th>( p_1 = 1.2 )</th>
<th>( p_1 = 2.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present iRBF 2D</td>
<td>3.225</td>
<td>8.836</td>
<td>5.530</td>
<td>3.309</td>
</tr>
<tr>
<td>Present iRBF 3D</td>
<td>3.337</td>
<td>8.671</td>
<td>5.472</td>
<td>3.282</td>
</tr>
<tr>
<td>Ho et al. [82], CS-FEM</td>
<td>3.301</td>
<td>8.748</td>
<td>5.504</td>
<td>3.302</td>
</tr>
<tr>
<td>Garcea et al. [231], FEM</td>
<td>3.280</td>
<td>8.718</td>
<td>5.467</td>
<td>3.280</td>
</tr>
<tr>
<td>Nguyen-Xuan et al. [81], ES-FEM</td>
<td>3.297</td>
<td>8.722</td>
<td>5.493</td>
<td>3.296</td>
</tr>
</tbody>
</table>
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Table 5.6: Symmetric continuous beam: shakedown load factors

<table>
<thead>
<tr>
<th>Authors</th>
<th>Loading cases</th>
<th>1.2 ≤ p₁ ≤ 2</th>
<th>0 ≤ p₁ ≤ 2</th>
<th>0 ≤ p₁ ≤ 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0 ≤ p₂ ≤ 1</td>
<td>0.6 ≤ p₂ ≤ 1</td>
<td>0 ≤ p₂ ≤ 1</td>
</tr>
<tr>
<td>Present iRBF 2D</td>
<td>3.217</td>
<td>2.333</td>
<td>2.308</td>
<td></td>
</tr>
<tr>
<td>Present iRBF 3D</td>
<td>3.228</td>
<td>2.357</td>
<td>2.276</td>
<td></td>
</tr>
<tr>
<td>Ho et al. [82], CS-FEM</td>
<td>3.362</td>
<td>2.228</td>
<td>2.205</td>
<td></td>
</tr>
<tr>
<td>Chen et al. [90], EFG</td>
<td>3.297</td>
<td>2.174</td>
<td>2.152</td>
<td></td>
</tr>
<tr>
<td>Garcea et al. [231], FEM</td>
<td>3.244</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Nguyen-Xuan et al. [81], ES-FEM</td>
<td>3.259</td>
<td>2.036</td>
<td>2.016</td>
<td></td>
</tr>
</tbody>
</table>

Figure 5.19: Symmetric continuous beam: stress fields in case of \([p₁, p₂] = [2, 0]\)

Figure 5.20: Symmetric continuous beam: stress fields in case of \([p₁, p₂] = [0, 1]\)
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Figure 5.21: Symmetric continuous beam: stress fields in case of $[p_1, p_2] = [1.2, 1]$

Figure 5.22: Symmetric continuous beam: stress fields in case of $[p_1, p_2] = [2, 1]$

Figure 5.23: Continuous beam: iRBF load domains compared with other methods
The approximated limit and shakedown load domains for all load range $p_N \in [-1, 1]$ and $p_M \in [-1, 1]$ are compared with those using smoothed finite elements in Figure 5.23. Stress fields for various load cases are also plotted in Figures 5.19 – 5.22.

### 5.3.5 A simple frame with different boundary conditions

![Figure 5.24: A simple frame: geometry, loading, boundary conditions](image)

(a) Model A  
(b) Model B

Figure 5.24: A simple frame: geometry, loading, boundary conditions

![Figure 5.25: A simple frame: nodal mesh](image)

(a) Nodal discretization 2D  
(b) Nodal discretization 3D

Figure 5.25: A simple frame: nodal mesh

The last example is the simple frames with different boundary conditions, see Figure 5.24 (all dimensions in cm). The problem are investigated in [82, 231] under plane stress condition. Data for analysis is given as: $E = 2 \times 10^5$ MPa, $\nu = 0.3$, $\ldots$. 

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Table 5.7: A simple frame (model A): limit and shakedown load multipliers

<table>
<thead>
<tr>
<th>Authors</th>
<th>Limit analysis</th>
<th>Shakedown analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p_1 = 0.4$</td>
<td>$0.4 \leq p_1 \leq 1.0$</td>
</tr>
<tr>
<td></td>
<td>$p_2 = 3.0$</td>
<td>$1.2 \leq p_2 \leq 3.0$</td>
</tr>
<tr>
<td>Present iRBF 2D</td>
<td>3.153</td>
<td>2.649</td>
</tr>
<tr>
<td>Present iRBF 3D</td>
<td>3.261</td>
<td>2.676</td>
</tr>
<tr>
<td>Ho et al. [82], CS-FEM</td>
<td>2.981</td>
<td>2.452</td>
</tr>
<tr>
<td>Garcea et al. [231], FEM</td>
<td>2.831</td>
<td>2.473</td>
</tr>
</tbody>
</table>

Table 5.8: A simple frame (model B): limit and shakedown load multipliers

<table>
<thead>
<tr>
<th>Authors</th>
<th>Limit analysis</th>
<th>Shakedown analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p_1 = 0.4$</td>
<td>$0.4 \leq p_1 \leq 1.0$</td>
</tr>
<tr>
<td></td>
<td>$p_2 = 3.0$</td>
<td>$1.2 \leq p_2 \leq 3.0$</td>
</tr>
<tr>
<td>Present iRBF 2D</td>
<td>4.152</td>
<td>3.964</td>
</tr>
<tr>
<td>Present iRBF 3D</td>
<td>4.209</td>
<td>4.172</td>
</tr>
<tr>
<td>Ho et al. [82], CS-FEM</td>
<td>4.186</td>
<td>3.817</td>
</tr>
<tr>
<td>Garcea et al. [231], FEM</td>
<td>4.207</td>
<td>3.925</td>
</tr>
</tbody>
</table>

Figure 5.26: Simple frame (model A): stress fields in case of $[p_1, p_2] = [3, 0.4]$  
(a) Elastic stress field  
(b) Residual stress field  
(c) Limit stress field

Figure 5.27: Simple frame (model A): stress fields in case of $[p_1, p_2] = [1.2, 1]$  
(a) Elastic stress field  
(b) Residual stress field  
(c) Limit stress field

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(a) Elastic stress field  (b) Residual stress field  (c) Limit stress field

Figure 5.28: Simple frame (model A): stress fields in case of $[p_1, p_2] = [3, 1]$ 

(a) Elastic stress field  (b) Residual stress field  (c) Limit stress field

Figure 5.29: Simple frame (model B): stress fields in case of $[p_1, p_2] = [3, 0.4]$ 

(a) Elastic stress field  (b) Residual stress field  (c) Limit stress field

Figure 5.30: Simple frame (model B): stress fields in case of $[p_1, p_2] = [1.2, 1]$ 

(a) Elastic stress field  (b) Residual stress field  (c) Limit stress field

Figure 5.31: Simple frame (model B): stress fields in case of $[p_1, p_2] = [3, 1]$
\( \sigma_p = 10 \text{ MPa} \) and the thick of frame \( t = 10 \text{ cm} \). The frames are subjected to biaxial load with the loading domain defined as

\[
0.4 \leq p_1 \leq 1; \quad 1.2 \leq p_2 \leq 3
\]

Numerical solutions obtained using the present iRBF-based equilibrium formulation are reported in Table 5.7 and 5.8. The approximated limit and shakedown load domains for different models are compared with those using smoothed finite elements [82] in Figure 5.32, showing excellent agreement. Stress fields for various load cases are also plotted in Figures 5.26 – 5.31.

### 5.4 Conclusions

A quasi-static approach based on integrated radial basis function mesh-free method and conic programming is proposed for direct analysis of structures. Instead of approximating stress fields, in the present formulation the virtual displacement fields are approximated by stabilized iRBF shape functions, and equilibrium condition for residual stress are satisfied in its weak form by using the virtual work principle. With the use of stabilized iRBF shape functions, equilibrium equations and yield conditions are enforced at discretized nodes, keeping the size of the resulting optimization problem to be minimum. Numerical examples show that the present requires less optimization CPU time comparing with other shakedown al-
Algorithms in the literature, and has a faster convergence behavior in comparison with the RPIM approach. The proposed approach is capable of providing solutions rapidly, and hence load domains of 2D and 3D structures can be approximated efficiently. Moreover, the present method is able to capture stress fields at limit state for various problems.
Chapter 6

Kinematic yield design computational homogenization of micro-structures using the stabilized iRBF mesh-free method

6.1 Introduction

The applications of iRBF-based meshless method for direct analysis of different structures subjected to various loading conditions have been presented in chapters 3, 4 and 5. In current chapter, the stabilized iRBF formulation previously developed in chapter 5 will be employed for limit analysis of microstructures.

As the increasing use of composite and heterogeneous materials in practical engineering structures, the estimation of their effective properties plays a vital role in safety assessment as well as structural design. The elastic-plastic incremental method can be employed to predict the ultimate load and collapse mechanism of structures. However, direct method, e.g., limit analysis shows more effectively, i.e. the critical status of structures can be determined without any knowledge of whole loading path history [9, 237]. The first theoretical framework of limit analysis combined with homogenization technique for computation of heterogeneous microstructures was introduced in [115–117]. The numerical formulations using various mathematical solvers were developed then, for instance finite element method and linear algorithms [118], static direct methods and interior point algorithms [25, 119, 120] or kinematic formulations in combination with nonlinear programming [121, 125]. Recently, with the use of finite element method and second order cone programming, a numerical procedure based on the combination of limit analysis and homogenization

\[\text{1\footnote{based on P. L. H. Ho, C. V. Le, and Phuong H. Nguyen. “Kinematic yield design computational homogenization of micro-structures using the stabilized iRBF mesh-free method,” Applied Mathematical Modelling, submitted on Feb 2020.}}\]
theory for periodic materials was proposed in [126].

This study aims to develop a novel computational homogenization approach for upper bound limit analysis of microstructures using iRBF meshless method. The stability conforming nodal integration (SCNI) technique proposed in [166] is utilized to improve the performance of proposed approach. In addition, the plastic dissipation will be transformed into the form of a sum of norms and the resulting optimization are then formulated as conic one. The benchmark numerical examples will be considered and the good agreement in comparison to previous procedures proves the performance of present method.

6.2 Limit analysis based on homogenization theory

Consider a heterogeneous representative volume element \( \Omega \in \mathbb{R}^2 \) at every material point \( \mathbf{x} \) in the heterogeneous macroscopic-continuum \( V \in \mathbb{R}^2 \). The microstructure is subjected to the body force \( \mathbf{f} \), the surface load \( \mathbf{t} \) on the static boundary \( \Gamma_t \) and fixed by the displacement field \( \mathbf{u} \) on the kinematic boundary \( \Gamma_u \). Assuming that all constitutions of ductile composite are rigid-perfectly plastic and the strain of constitutions obey the normality rule. The kinematic approach in framework of limit analysis for computation homogenization described in [121, 125, 183] will be taken into account in this study.

It is important to note that most of yield criterion can be expressed in the following form

\[
\psi(\sigma) = \sqrt{\sigma^T \mathbf{P} \sigma} - 1 \tag{6.1}
\]

where \( \mathbf{P} \) is the coefficient matrix consisting strength properties of materials. For orthotropic materials, Hill’s criterion is often used; and \( \mathbf{P} \) for plane stress problem is given by

\[
\mathbf{P} = \begin{bmatrix}
\kappa_{zz} + \kappa_{xy} & -\kappa_{xy} & 0 \\
-\kappa_{xy} & \kappa_{xy} + \kappa_{yz} & 0 \\
0 & 0 & 3\eta_{xy}
\end{bmatrix} \tag{6.2}
\]
where the constants of material features can be expressed by

\[
\begin{align*}
\kappa_{xx} &= \frac{1}{2} \left( \frac{1}{\sigma_{px}^2} + \frac{1}{\sigma_{py}^2} - \frac{1}{\sigma_{pz}^2} \right); & \kappa_{xy} &= \frac{1}{2} \left( \frac{1}{\sigma_{px}^2} + \frac{1}{\sigma_{py}^2} - \frac{1}{\sigma_{pz}^2} \right) \\
\kappa_{yz} &= \frac{1}{2} \left( \frac{1}{\sigma_{py}^2} + \frac{1}{\sigma_{pz}^2} - \frac{1}{\sigma_{px}^2} \right); & \eta_{xy} &= \frac{1}{3\tau_{pxy}^2}
\end{align*}
\] (6.3a)

(6.3b)

with \((\sigma_{px}, \sigma_{py}, \sigma_{pz})\) are the uniaxial yield stresses related to three orthotropic dimensions \((x, y, z)\); and \(\tau_{xy}\) is the shear yield stress of materials.

For isotropic materials, the so-called von Mises criterion, a special case of Hill’s criterion when \(\sigma_{px} = \sigma_{py} = \sigma_{pz} = \sigma_p\) and \(\tau_{pxy} = \frac{\sigma_p}{\sqrt{3}}\), is frequently applied; and matrix \(P\) for plane stress problem is given by

\[
P = \frac{1}{\sigma_p} \begin{bmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}
\] (6.4)

In framework of limit analysis, the strain rates are assumed to obey the normality rule; and therefore, the power of dissipation can be formulated in term of strain rates as

\[
D(\epsilon) = \int_\Omega \sqrt{\epsilon^T \Theta \epsilon} \, d\Omega = \int_\Omega \sqrt{(E + \tilde{\epsilon})^T \Theta (E + \tilde{\epsilon})} \, d\Omega
\] (6.5)

where \(\Theta\) is the inversion matrix of \(P\).

Omitting the body force \(f\) and applying the principle of microscopic virtual work (2.58), the normalization condition of external power can be rewritten as

\[
F(u) = \int_{\Gamma_t} t^T u d\Gamma = \Sigma^T E = 1
\] (6.6)

where \(\Sigma^T\) and \(E\) are the overall stress and strain.

Now, the kinematic limit formulation of computational homogenization analysis for a periodic micro-structure can be expressed as

\[
\lambda^+ = \min \int_\Omega \sqrt{(E + \tilde{\epsilon})^T \Theta (E + \tilde{\epsilon})} \, d\Omega
\] (6.7a)

s.t \[
\begin{align*}
\Sigma^T E &= 1 \\
\bar{u} \text{ periodic on } \Gamma_u
\end{align*}
\] (6.7b)
The upper-bound of macroscopic limit strength \( \lambda^+ \Sigma \) can be determined by solving the nonlinear problem (6.7). The main difference of (6.7) in comparison with the formulation of kinematic limit analysis for structure is in the periodic boundary condition. Furthermore, it should be noted that present study only considers the continuous velocity fields; and if the velocity fields are assumed to be discontinuous, the dissipated power generated by discontinuities must be taken into account.

### 6.3 Discrete formulation using iRBF method

Following the homogenization theory, all variables related to the microscopic structures are split to two parts: mean fields averaged over RVE and fluctuation fields. In the kinematic formulation, the local microscopic fluctuation strain \( \tilde{\epsilon}(x) \) at point \( x \) can be calculated via the derivative of the fluctuation displacement \( \tilde{u}(x) \) approximated using iRBF method as

\[
\tilde{u}^h(x) = \sum_{i=1}^{N} \Phi_i(x) \tilde{u}_i = N\mathbf{d} \tag{6.8a}
\]

\[
\tilde{\epsilon}(x) = \sum_{i=1}^{N} \tilde{\Phi}_{i,\alpha}(x) \tilde{u}_i = \mathbf{Bd} \tag{6.8b}
\]

where \( N \) is number of nodes scattered within problem domain; \( \mathbf{N} \) denotes the iRBF shape function; \( \mathbf{B} \) is the strain-displacement matrix consisting the smoothed version of shape function derivatives; and \( \mathbf{d} \) is the nodal fluctuation displacement vector.

With the use of SCNI technique for the numerical integration, the plastic dissipation well-known as the objective function of the optimization problems can be expressed as

\[
D_p(\epsilon) = \sum_{i=1}^{N} \sigma_p A_i \sqrt{(E + B_i \mathbf{d})^T \Theta (E + B_i \mathbf{d})} \tag{6.9}
\]

where \( \sigma_p \) is the yield stress of material, \( A_i \) is the area of the \( i^{th} \) nodal representative domain created using Voronoi diagrams.

In this study, the optimization problem will be formulated in form of second order cone programming (SOCP) ensuring it can be solved using the highly efficient solves. Hence, a form of sum of norms can be used to calculate the internal
dissipation power as
\[ D_p(\epsilon) = \sum_{i=1}^{N} \sigma_p A_i \| \rho_i \| \] (6.10)

where \( \| \cdot \| \) denotes the Euclidean norm and \( \rho \) is the vector of additional variables defined by
\[ \rho_i = Q^T (E + B_i d) \] (6.11)

with \( Q \) denotes the Cholesky factor of \( \Theta \).

Next, the periodic feature of the fluctuation displacement for nodes on the boundary of RVE needs to be enforced. Denoting \( \Gamma^+ \) and \( \Gamma^- \) for the positive and negative boundaries such that \( \Gamma^+ \cup \Gamma^- = \Gamma \) and \( \Gamma^+ \cap \Gamma^- = \emptyset \), the periodic condition for every pair of points \( \{x^+, x^-\} \) on two opposite boundaries can be expressed as
\[ \tilde{u}(x^+) - \tilde{u}(x^-) = 0 \] (6.12)

Assembling to the global matrix, equation (6.12) can be rewritten as
\[ Cd = 0 \] (6.13)

Finally, by introducing the auxiliary variables \( (t_1, t_2, \ldots, t_N) \), the optimization problem can be formulated in form of conic programming as follows
\[ \lambda^+ = \min \sum_{i=1}^{N} \sigma_p A_i \| \rho_i \| \] (6.14a)
\[ \begin{cases} 
\Sigma^T E = 1 \\
C d = 0 \\
\| \rho_i \| \leq t_i, \quad i = 1, 2, \ldots, N 
\end{cases} \] (6.14b)

The numerical implementation of the proposed approach is shown in Figure 6.1.

### 6.4 Numerical examples

In this section, various benchmark problems of computational homogenization for limit analysis, in which the numerical solutions are available, will be investigated to illustrate the performance of proposed method. A square RVE of \( a \times a = 1 \times 1 \)
mm, and the shape parameters ($\alpha_s = 0.00001$, $\beta_s = 3$) are used for all examples. The plane stress model is assumed and number of variables $N_{\text{var}}$ in the problem is equal to $6 \times N + 3$. The resultant optimization problems are solved using the commercial software package Mosek on a 2.8 GHz Intel Core i7 PC running Windows 10.
6.4.1 Perforated materials

Estimating load-bearing capacity of perforated materials treated as a special composite plays an important role in engineering structural design. In this example, two perforated material models including a rectangular and a circle hole at center are considered. The RVE is subjected to an orthogonal macroscopic stress \((\Sigma_{11}, \Sigma_{22})\) in plane as shown in Figure 6.2, where \(\theta\) is the angle between the principle stress and \(x\)-axis. The matrix materials and yield stresses of rectangular and circular hole RVEs are summarized in Table 6.1, and the behavior of both material models is assumed to obey the von Mises yield criterion. Figure 6.3 illustrates the scheme of nodal discretization using Voronoi diagram. The problems have been investigated using kinematic formulation in \([121, 122, 126]\) and quasi-static formulation in \([120]\).

![Figure 6.2: RVEs of perforated materials: geometry, loading and dimension](image)

**Table 6.1: Perforated materials: the given data**

<table>
<thead>
<tr>
<th>Material models</th>
<th>Matrix material</th>
<th>Yield stress (\sigma_p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RVE with rectangular hole</td>
<td>Aluminium ((Al))</td>
<td>137 MPa</td>
</tr>
<tr>
<td>RVE with circular hole</td>
<td>Mild steel ((St3s))</td>
<td>273 MPa</td>
</tr>
</tbody>
</table>

In case of rectangular hole RVE, the problem is considered with different sizes of hole: \((L_1 \times L_2 = 0.1 \times 0.5 \text{ mm})\) and \((L_1 \times L_2 = 0.1 \times 0.7 \text{ mm})\). Table 6.2 shows the numerical solutions using iRBF procedure in comparison with those in \([122, 126]\). From the table, it can be observed that present method provides the highly-accurate solution with low computational cost. Number of variables in present formulation is

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less than those used in [126], while the numerical result is approximate. Moreover, taking advantage of the cone-based algorithm, present resultant optimization problem can be solved rapidly, the CPU-time taken in whole solving process is much lower than those in [122] using iterative algorithm. The good agreement of present solutions compared with previous results reported in [122, 126] using numerical as well as experimental procedures is expressed one more time in Figure 6.4 where the macroscopic uniaxial strength $\Sigma_{11}$ for the case of loading angle $\theta = 0^\circ$ corresponding to two different sizes of rectangular hole are plotted. In addition, it can be seen from both sub-figures that the upper-bound solutions obtained using present iRBF approach are slightly lower (better) than available those in other studies.

![Figure 6.3: RVEs of perforated materials: nodal discretization using Voronoi cells](image_url)

**Table 6.2: Rectangular hole RVE ($L_1 \times L_2 = 0.1 \times 0.5$ mm)**

<table>
<thead>
<tr>
<th>Author and approach</th>
<th>$\Sigma_{11}/\sigma_p$</th>
<th>$sdof$</th>
<th>CPU-Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present study, iRBF</td>
<td>0.5591</td>
<td>5246</td>
<td>10</td>
</tr>
<tr>
<td>Li et al. [122], FEM</td>
<td>0.5600</td>
<td>1920</td>
<td>95</td>
</tr>
<tr>
<td>Le et al. [126], FEM</td>
<td>0.5561</td>
<td>8140</td>
<td>6</td>
</tr>
</tbody>
</table>

$sdof$ denotes the total system degrees of freedom

Next, the effect of microscopic hole on the overall strength of perforated materials is investigated. The RVE with circular hole is considered with various variable dimensions of perforation and loading angles. The uniaxial macroscopic strengths
\( \Sigma_{11} \) and the limit strength domains in plane \((x_1, x_2)\) are plotted in Figures 6.5 and 6.6, respectively. Obviously, from Figure 6.6 it is reasonable that the effective macroscopic strength decreases when increasing ratio \(R/a\). Again, present solutions well agree with available those in [121, 126], see Figure 6.5.

Figure 6.4: Rectangular hole RVE: limit uniaxial strength \( \Sigma_{11} \) in comparison with other procedures

Figure 6.5: Circular hole RVE: limit uniaxial strength \( \Sigma_{11} \) in comparison with other procedures
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Figure 6.6: Circular hole RVE: limit macroscopic strength domain with different values of fraction $R/a$ and loading angle $\theta$

(a) $\theta = 0^\circ$

(b) $\theta = 45^\circ$

Figure 6.7: Perforated materials: macroscopic strength domain at limit state

(a) Rectangular hole RVE

(b) Circular hole RVE
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Figure 6.8: Rectangular hole RVE \((L_1 \times L_2 = 0.1 \times 0.5 \text{ mm})\): the distribution of plastic dissipation

\[
\begin{align*}
&\text{(a)} \quad \theta = 0^\circ \\
&\text{(b)} \quad \theta = 45^\circ \\
&\text{(c)} \quad \theta = 90^\circ
\end{align*}
\]

Figure 6.9: Rectangular hole RVE: macroscopic strength domain under three-dimensions loads \((\Sigma_{11}, \Sigma_{12}, \Sigma_{22})\)

(a) 3D view

(b) \(\Sigma_{11} - \Sigma_{22}\) view

(c) \(\Sigma_{11} - \Sigma_{12}\) view

(d) \(\Sigma_{22} - \Sigma_{12}\) view

In addition, the approximate macroscopic strength domain of perforated materials with rectangular hole \((L_1 \times L_2 = 0.1 \times 0.7 \text{ mm})\) and circle hole \((R = 0.25 \times a)\) for angles of \(\theta = 0^\circ\) and \(\theta = 45^\circ\) are illustrated in Figure 6.7. The strength in \(\Sigma_{22}\) direction is greater than that in \(\Sigma_{11}\) direction in case of rectangular hole RVE, whereas those are equivalent in case of circular hole RVE. Figures 6.9 and 6.11 plot
the macroscopic strength domain in limit state of both perforated models under three-dimensions of applied loading ($\Sigma_{11}, \Sigma_{22}, \Sigma_{12}$). The plastic dissipation distributions representing the failure mechanism of RVEs are also presented in Figures 6.8 and 6.10.

Figure 6.10: Circular hole RVE ($R = 0.25 \times a$): the distribution of plastic dissipation

Figure 6.11: Circular hole RVE: macroscopic strength domain under three-dimensions loads ($\Sigma_{11}, \Sigma_{12}, \Sigma_{22}$)
6.4.2 Metal with cavities

![Diagram of metal sheet with cavities](image)

Figure 6.12: Metal sheet with cavities: geometry and loading

![Nodal discretization and macroscopic strength domain](image)

Figure 6.13: Metal with cavities: nodal discretization and macroscopic strength domain

This example examines a perforated metal sheet with $4 \times 4$ holes regularly arranged in plane $(x_1, x_2)$ as seen in Figure 6.12. The sheet of metal has dimension $a \times a$, the radius of holes and the distances are chosen such that $L/a = 0.2$ and $R/a = 0.05$. The square pattern is subjected to a set of orthogonal load $(\Sigma_{11}, \Sigma_{22})$,
in which the angle made by $\Sigma_{11}$ and $x_1$-axis is $\theta$. Assuming that material obeys the von Mises criterion. The nodal discretization of computational domain is presented in Figure 6.13(a).

Figure 6.14: Metal with cavities: macroscopic strength domain under three-dimensions loads ($\Sigma_{11}, \Sigma_{12}, \Sigma_{22}$)

Figure 6.15: Metal with cavities: the distribution of plastic dissipation

The problem has been study in [126], and the comparison of the approximate macroscopic strength domain obtained by iRBF method and those reported in [126]
is expressed in Figure 6.13(b). From the figure, it can be observed that present solutions well agree with results in [126]. The limit domain of macroscopic strength for case of three dimension of applied load ($\Sigma_{11}, \Sigma_{22}, \Sigma_{12}$) are presented in Figure 6.14. The distribution of plastic dissipation corresponding to different values of loading angle $\theta$ are also plotted in Figure 6.15. It is easy to see that the yield zone makes the lines connecting holes and the boundary of RVE.

6.4.3 Perforated material with different arrangement of holes

This section considers a perforated material with two circular holes arranged such that the line connecting their centroid forms with $x_1$ axis an angle $\varphi$, and the effect of hole’s arrangement on the bearing capacity of macro-structures is investigated. The micro-structure is subjected to the macroscopic tensile load as Figure 6.16.

![Perforated material with two hole: geometry and loading](image)

Figure 6.16: Perforated material with two hole: geometry and loading

Denoting $V_f$ for the volume fraction of void, the centroid’s coordinates of holes and their radius can be determined by

$$x_{1i} = \pm (0.05 + R) \cos \varphi, \quad i=1,2$$
$$y_{1i} = \pm (0.05 + R) \sin \varphi, \quad i=1,2$$

and $R = \sqrt{\frac{V_f}{2\pi}}$ (6.17)

Various values of angle $\varphi$ for different volume of voids are studied. Figure 6.17 clearly illustrates the decrease of macroscopic tensile strength when increasing the angle $\varphi$ for all cases of $V_f$. The numerical solutions are also compared with those reported in [126], and the agreement of solutions demonstrates the resonability of present results. The plastic dissipation distribution for the volume fraction $V_f = 0.2$
are plotted in Figure 6.18. The results point out that the failure mechanism as well as the strength of macro-structures is significantly affected by the change of the void’s location in micro-scale level.

Figure 6.17: Perforated material with two hole: the comparison of macroscopic strengths obtained using iRBF and FEM in [126]

Figure 6.18: Perforated material with two hole: the distribution of plastic dissipation

6.5 Conclusions

The plastic limit strength and the collapse mechanism of materials has been studied using the combination of direct analysis and homogenization theory. By
means of second-order cone programming and the iRBF approximation, the result- 
ing optimization problems are kept in minimum size and solved rapidly. The good agreement of numerical solutions in comparison with other studies shows the computational efficiency of proposed method. In future work, the plane strain or three-dimensions problems, in where the volumetric locking phenomena is required to be handled, are extended. In addition, more complicate effects, e.g. material interfaces, multiple crack or even variable loading should be considered.
Chapter 7

Discussions, conclusions and future work

In this thesis, a novel numerical method employing the combination of the integrated radial basis function-based mesh-free method and second-order cone programming is developed for limit and shakedown analysis. The application of proposed approach for various problems regarding to structures and materials has been investigated in chapters 3, 4, 5, and 6. The current chapter expresses several discussions on the major issues arising during the course of the research, thereby both advantages and disadvantages of present procedure are outlined. Then, the contributions of this study are summarized, and several suggestions for future work are recommended.

7.1 Discussions

7.2 The convergence and reliability of obtained solutions

In this thesis, the iRBF method is developed for both displacement and equilibrium formulations of direct analysis, for which the strictly lower-bound and upper-bound models are applied in chapters 3, 4, and 6 whereas the quasi-lower bound one is employed in chapter 5.

Theoretically, for all models used, the numerical solutions converge to the exact value when increasing the nodal distribution, but using different formulations, the convergence behaviour is different as illustrated in Figure 7.1. Usually, the upper bound solutions approach to the actual collapse load multiplier from above, while the lower bound results converge from bellow as seen in chapters 3 and 4. However, owing to the approximation of displacement field and equilibrium equations are satisfied in a weak form, the equilibrium formulation in chapter 5 provides solu-
Discussions, conclusions and future work

It is interesting to note that although the analytical solutions are not available for almost practical engineering problems, the mean values of numerical solutions independently obtained using reliable upper bound and lower bound estimations can be recommended as actual safety loads for the use in structural design.

![Convergent study (Prandtl’s problem in chapters 3 and 5)](image)

The numerical solutions of limit and shakedown analysis for various benchmark problems using the iRBF method with 2D and 3D discretizations have been reported in this thesis. In several problems, e.g. in sections 3.3.1, 4.3.1, 5.3.1 and 5.3.2 present solutions are priorly compared to the available analytical those in the literature in order to examine the accuracy of numerical results. It is important to note that, all relative errors are less than 1%, especially in sections 4.3.1 and 5.3.2 these errors are approximate 0%. The good agreement of proposed method in comparison with other studies, in which both analytical and numerical schemes are used, demonstrates the reliability of the approach.

7.2.1 The advantages of present method

As clarified, the most attractive feature of direct methods is the capability to assess the ultimate state of structures or materials without the step-by-step analysis, and not only safety load multipliers but also collapse mechanics of the structural systems are determined effectively. The positives of proposed method are offered by
a robust numerical approximation scheme and a powerful mathematical tool, that will be discussed following.

For one thing, there is the matter of the mesh in numerical discretization. As known, physical problems are usually expressed in terms of partial differential equations (PDEs), but that cannot be solved with analytical approach in almost situations. As a result, PDEs need to be treated using approximations typically based upon different types of discretizations. Traditionally, a favorite scheme, which is commonly employed in mesh-base methods, e.g. FEM, FVM or BEM, is that the physical domain is subdivided into a finite discrete elements connected together at nodes. The priori definition of nodal connectivity is known as the mesh permitting the compatibility of the interpolation and making the resulting approximation be continuous. In general, mesh-based methods are robust particularly for mechanical analysis; however, the creation of the mesh may become the major one of the total cost in whole process of the numerical implementation. On contrary, from the practical standpoint, the absence of the mesh in iRBF method is an attractive feature, decreasing the computational cost of such problems.

One more competitive advantage of iRBF method in comparison with mesh-based ones as well as other mesh-free schemes gains from the shape function. In traditional numerical procedures as FEM, the shape function of a node has low-order and only affects on elements attached to it, while in the meshless methods, the shape function can be flexibly constructed, i.e. the ability to create overlapping nodal influent domain increasing the continuity of approximation, or the initiative in choosing the order of functions. In mesh-free methods, the high-order shape function not only help to provide the highly accurate solution with rapid convergent rate, but also gives the effective treatment for the volumetric locking phenomena in solid mechanics problems. In this study, the iRBF method is constructed based on the multiple integration. As a result, its order is higher than other meshless ones, e.g. RPIM when using similar basis function. The better performance of present method compared to RPIM has been reported in section 5.3.1, i.e., iRBF method takes less CPU-time to provide more accurate and rapidly convergent results than RPIM when using the same nodal distribution.

Moreover, with the use of iRBF method in combination with second order cone programming and collocation procedure, number of variables required for the resulting optimization problems can be reduced significantly. It is worth noting that
using present formulation, the optimization problems with thousand variables can be speedily solved in seconds. The advantage of proposed approach in terms of computational cost was clearly discussed in chapters 3, 4, 5, and 6. For instance, in chapter 4 an application to limit state analysis of reinforced concrete slabs has been presented. There is only transverse velocity in need to be approximated; therefore the total number of variables in the resultant optimization problem is kept minimum, i.e. equal to the number of discretized nodes in the problem domain. In section 4.3.1 present method was compared with a finite element method named CS-HCT and a mesh-free one so-called EFG. Clearly, the solutions from three methods converge to similar values, but generally, iRBF formulation shows more efficiently than those of both CPU-time and number of variables.

In addition, the advantage of the iRBF method over the EFG and almost mesh-free approaches is that iRBF shape function satisfies Kronecker-delta property; and therefore, there is no need of any special treatment when enforcing boundary conditions.

Last but not least is the improvement of iRBF method itself using stabilized conforming nodal integration technique. The collocation method is employed in this thesis, meaning that all conditions are directly satisfied at discretized nodes in problem domain. However, in chapter chapters 3 and 4 with the use of classical iRBF formulation, the global nodal influent domain is employed to ensure the accuracy of solutions, resulting in dense matrices included in optimization problems. Whereas, using iRBF method combined with SCNI scheme, which is so-called stabilized iRBF, in chapters 5 and 6 only the local domain is required in the approximation, and those matrices become sparse, decreasing the computer memory and CPU-time in solving process significantly. The advantage of stabilized iRBF in comparison with classical iRBF is clearly pointed out in section 5.3.1. It can be observed that solutions obtained from both models of iRBF method converge to the analytical one, but the local iRBF formulation with the support of SCNI provides the improvement of computational efficiency concerning accuracy and time taken for resulting optimization problem when using similar basis (nodal distribution and shape parameter).
7.2.2 The disadvantages of present method

The advantages and disadvantages of iRBF method are generated from the key difference of mesh-free procedures and mesh-based ones, i.e. the shape function and strategy to construct it. In previous section, the positive features of proposed method are discussed, now the negative those will be focused on.

Generally, the high-order shape function makes iRBF method as well as mesh-free ones more advantageous compared to mesh-base procedures to provide the accurate solutions with the speed convergence. The only disadvantage here is that it takes more computational run-time, and thus the cost of these process is still high. However, that is not the major obstacle of this study since the biggest challenge of direct analysis is dealing with the optimization problems accounting the most of overall cost. Employing the combination of stabilized iRBF approach and primal-dual interior-point SOCP algorithm, the resulting formulation is kept in minimum size and then solved rapidly using the highly efficient solvers.

Another limitation of meshless methods preventing the acceptance of the engineering community is that there are several factors affecting on the accuracy of outcomes must be priorly selected, for instance, the influence domain size or the coefficients of the shape function. A set of factors for one case may not work correctly for another ones, i.e., in case of elastic analysis, the dimensionless parameters should be chosen as \((\alpha_s = 10^{-5} \div 2.5, \beta_s = 3)\); whereas in direct analysis, those values are given as \((\alpha_s = 10^{-5} \div 2, \beta_s = 6)\) for structure scale and \((\alpha_s = 10^{-5}, \beta_s = 3)\) for material scale. It is difficult to find out a unified standard of approximation properties for all practical problems. In literature, the gap for choice is determined by trial and error.

In short, besides many advantages of computational aspect, there are several matters needed to be overcome in iRBF as well as other mesh-free approximation techniques. However, it should be realized that mesh-free methods are still in their infancy. They are being continuously improved to be integrated into commercial software packages for structural design.
7.3 Conclusions

The major objective of the research is to develop a robust numerical approach for direct analysis of structures and materials widely used in practical engineering, resulting in the use of iRBF-based meshless procedure and the primal-dual second order cone programming algorithm in the thesis. Detailed conclusions on specific problems are presented in chapters 3, 4, 5 and 6. This chapter will give the outline of the remarkable points following.

Firstly, this is the first time the iRBF mesh-free method combined with stability conforming nodal integration (SCNI) is developed to deal with the problems in the area of limit and shakedown analysis of structures and materials. The iRBF shape function is used to approximate the displacement as well as stress fields. The advantages of such procedure can be summarized as follows

- Unlike the traditional iRBF approach, for which the constraints in problems are imposed at various Gauss points, using the collocation method and SCNI scheme in this study, the kinematic and equilibrium conditions as well as the numerical integration in resulting optimization problems can be directly applied at scattered nodes, making proposed method truly mesh-free and reducing the size of formulated problems.

- The high-order iRBF shape functions are constructed on the overlapping influence domains, thus the enforcement of discontinuous condition at the interfaces of neighbour computational cells is not necessary.

- The high-order iRBF shape functions help to keep total number of variables to a minimum, i.e. for the kinematic discretization of bending slabs in chapter 4, only one degree of freedom (deflection) needs to be approximated instead of three those (deflection and two rotations) as in finite element method.

- The shape function satisfies Kronecker-delta property, which is absent in almost meshless procedures. As a result, the essential boundary conditions in problems can be similarly imposed as in finite element formulation. This characteristic also makes the matrices spare, decreasing the CPU run-time in whole solving process.

Secondly, by mean of conic algorithm, the largest obstacle in direct analysis is
overcome. The optimization problems are cast as second order cone programming, and then solved using an efficient commercial software package named Mosek. The numerical examples investigated in the thesis show that with the use of primal-dual interior point algorithm, a problem with thousand variables can be solved in seconds, proving that present method can be applied for large scale problems in engineering practice.

In conclusion, the combination of stabilized iRBF-based mesh-free method and SOCP algorithm results in a robust numerical procedure for limit and shakedown analysis, for which not only the safety loads are rapidly determined, but also the collapse mechanics of structures and the yield surface of heterogeneous materials are effectively captured. The good agreement in comparison with the analytical approach as well as other numerical schemes in literature fully justifies the computational effects of proposed method.

### 7.4 Suggestions for future work

Although present research has met most of initial objectives, there are several issues needed to be overcome in future works, and there are some techniques are able to be applied to improve the computational aspect of proposed method. Following extensions are recommended for further development of present work.

As mentioned, in iRBF procedures as well as other mesh-free methods, the shape parameters and influent domain size impact on the accuracy and stability of outcomes significantly. However, those factors are variable, and there are not any common standard for choice yet. In most studies, the well-known strategy is selecting form a gap determining by trial and error. It is important to find out an efficient algorithm to optimize those values and discover an appropriate interval for almost problems. That is interesting topic to be taken into account in further studies.

Taking advantage of a positive feature of mesh-free methods, i.e. the absence of the mesh in numerical discretization, the adaptive technique, especially $h$-adaptivity, may be easily applied for the implementation of iRBF procedure. The natural conforming property of mesh-free approximations make the use of $h$-adaptivity, for which only nodes have to be added, comparably simple. Also, conceptually, the application of $p$-adaptivity in meshless methods is simpler than in mesh-based ones, there are only additional enrichment is required to be added to the basis function.
Based on a posterior error estimation, adaptive scheme will automatically refine at indicated locations, e.g. plastic zone, and leave out the refinement at other regions within the problem domain. Consequently, the convergence rate is improved, and the cost of computation can be reduced. It would be relevant to extend adaptive strategy to iRBF formulation for direct analysis of engineering structures.

Moreover, there are various practical engineering problems for which this thesis cannot cover, for instance, fracture problem - an interesting topic in solid mechanics. In fact, traditional methods as FEM are not well suit for the treatment of discontinuities which do not coincide to the original mesh line, leading to the development of the so-call eXtended Finite Element Method (XFEM) which is demonstrated to be an effective solution for crack problem. In further studies, an extension of enrich technique from XFEM to iRBF approximation could be a good idea for limit and shakedown analysis of fracture structures using iRBF approach.

Furthermore, in computational homogenization analysis of materials, only plane stress problems are investigated. In future works, the plane strain or three dimensions problems, where the volumetric locking needed to be handled, will be extended. The more complicate effects such as variable, cyclic or repeat loading, or even materials with diverse constitutes including material interfaces, multiple crack may be also considered in the problems.
List of publications

The results from parts of thesis have been presented at the national & international conferences and published in the domestic & international journals.

**International peer-reviewed journals**


**Domestic journals**


**International conferences**


National conferences


Bibliography


